Series Expansion for Two-Center Noninteger-n Coulomb Integrals*

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A series in the internuclear distance R is derived for the two-center Coulomb integral between noninteger-n Slater-type charge distributions. There are in general two types of terms: $R^{2N+\lambda}$ and $R^{n_1+n_2+3+N}$, ($N=0, 1, 2, \cdots$; $\lambda=|l_1-l_2|, |l_1-l_2|+1, \cdots, l_1+l_2$). Most of the coefficients are found from the coefficients of the overlap series by exploiting $\{\nabla^2$ (Coulomb integral) $=-4\pi$ (overlap integral)$\}$. When n_1 and n_2 are not integers but n_1+n_2 is, the series contains logarithmic terms. The series is for general values of n_1, n_2, l_1, l_2, m_1, m_2, ζ_1, ζ_2, and $\mathbf{R}$, and it converges for $R<\infty$.

INTRODUCTION

THE two-electron two-center six-dimensional Coulomb integral with noninterger-n Slater-type charge distributions was reduced to a one-dimensional integral by the Fourier-transform convolution theorem in a previous paper.[1] In this paper we derive a series expansion in the internuclear distance R for the final integration. As for the overlap-integral series (preceding paper[2]), the series is for general n_1, n_2, l_1, l_2, m_1, m_2, ζ_1, ζ_2, and $\mathbf{R}$, converges for $R<\infty$, and has logarithmic terms when $n_1+n_2=$ integer but $n_1\neq$ integer.

The series themselves are given by Eqs. (1), (20)–(23), (26), and (27).

FORMULATION

We start with Eq. (18) of I in the form

$$C_{n_1l_1m_1\zeta_1;n_2l_2m_2\zeta_2}(\mathbf{R})=M_{n_1l_1\zeta_1}M_{n_2l_2\zeta_2}\sum_{\lambda=|l_1-l_2|}^{l_1+l_2}8\pi^{3/2}(2\lambda+1)^{1/2}c^\lambda(l_2, m_2; l_1, m_1)\,Y_\lambda^{m_2-m_1}(\theta_R, \phi_R)\,\gamma^\lambda_{n_1l_1\zeta_1;n_2l_2\zeta_2}(R), \quad (1)$$

$$\gamma^\lambda_{n_1l_1\zeta_1;n_2l_2\zeta_2}(R)\equiv(16\pi^3)^{-1}(-1)^{l_1}i^\lambda\int_{-\infty}^{\infty}dk f_{n_1l_1\zeta_1}(k)f_{n_2l_2\zeta_2}(k)j_\lambda(kR). \quad (2)$$

All the symbols undefined above are defined in I. The series for $\gamma^\lambda(R)$ could be obtained by the same methods used for the corresponding $\mathcal{S}^\lambda(R)$ of the overlap integral,[2] but a much simpler procedure is to exploit the Poisson equation between Coulomb and overlap integrals[3]

$$\nabla^2C(\mathbf{R})=-4\pi S(\mathbf{R}), \quad (3)$$

which takes the form

$$[-R^{-1}(d/dR)^2R+\lambda(\lambda+1)R^{-2}]\gamma^\lambda(R)=\mathcal{S}^\lambda(R), \quad (4)$$

where $\mathcal{S}^\lambda(R)$ is defined by Eq. (6) of II, and which follows immediately from the spherical-Bessel equation[4a]

$$k^2j_\lambda(kR)=[-R(d/dR)^2R+\lambda(\lambda+1)R^{-2}]j_\lambda(kR). \quad (5)$$

DETAILS: $n_1+n_2\neq$INTEGER

Assume $n_1+n_2\neq$ integer and write

$$\gamma^\lambda(R)=\sum_N\gamma_N{}^\lambda R^N, \quad (6)$$

$$\mathcal{S}^\lambda(R)=\sum_N\mathcal{S}_N{}^\lambda R^N. \quad (7)$$

From Eq. (4), one obtains

$$\gamma_N{}^\lambda=[(\lambda-N)(\lambda+N+1)]^{-1}\mathcal{S}_{N-2}{}^\lambda, \quad (8)$$

which determines all the coefficients of the $\gamma(R)$ series from those of the $\mathcal{S}(R)$ series, except the coefficient $\gamma_\lambda{}^\lambda$, [since the lowest power of R in $\mathcal{S}^\lambda(R)$ is R^λ (see II)]. Except for $\lambda=|l_1-l_2|$, we use (for small R)[4b]

$$j_\lambda(kR)\sim2^\lambda\lambda!k^\lambda R^\lambda/(2\lambda+1)! \quad (9)$$

along with Eq. (2) and with Eq. (6) of II to obtain

$$\gamma_\lambda{}^\lambda=(1-4\lambda^2)^{-1}\mathcal{S}_{\lambda-2}{}^{\lambda-2}. \quad (10)$$

With one more coefficient, $\gamma_{|l_1-l_2|}{}^{|l_1-l_2|}$, and Eqs. (8) and (10), the entire series can be determined.

COEFFICIENT OF $R^{|l_1-l_2|}$

First assume $l_1>l_2$. [At the end we obtain the case $l_1<l_2$ by switching n_1, l_1, ζ_1 with n_2, l_2, ζ_2 and multiplying by $(-1)^{l_1+l_2}$; the $l_1=l_2=l$ case was given by Joy and Parr.[5]] After manipulations similar to those leading to

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[1] H. J. Silverstone, J. Chem. Phys. **45**, 4337 (1966), hereafter referred to as I.

[2] H. J. Silverstone, J. Chem. Phys. **46**, 4368 (1967), hereafter referred to as II.

[3] K. O-Ohata and K. Ruedenberg, J. Math. Phys. **7**, 539 (1966).

[4] *Handbook of Mathematical Functions*, M. Abramowitz and I. A. Stegun, Eds. (National Bureau of Standards, Washington, D.C., Appl. Math. Ser. No. 55, 1964): (a) p. 437, Eq. (10.1.1); (b) p. 437, Eq. (10.1.2); (c) p. 258, Eq. (6.2.1), (d) p. 263, Eq. (6.6.8); (e) p. 561, Eq. (15.4.1); (f) p. 559; (g) pp. 557–558.

[5] H. W. Joy and R. G. Parr, J. Chem. Phys. **28**, 448 (1958).

Eq. (11) of II, we obtain from Eqs. (2), (6), and (9), when $l_1>l_2$,

$$\gamma_{l_1-l_2}{}^{l_1-l_2}=-i(2\pi)^{-1}(-1)^{l_1}2^{l_1-l_2}(l_1-l_2)![(2l_1-2l_2+1)!]^{-1}\Gamma(n_1-l_1+1)\Gamma(n_2-l_2+1)$$
$$\times\int_\infty^{(\zeta_1^+)}dx\,x^{2l_1}\left[\left(x^{-1}\frac{d}{dx}\right)^{l_1}x^{-1}(\zeta_1-x)^{l_1-n_1-1}\right]\left[\left(x^{-1}\frac{d}{dx}\right)^{l_2}x^{-1}(\zeta_2+x)^{l_2-n_2-1}\right]. \quad (11)$$

The integration path (which is just the contour Λ_2 in Fig. 3 of II) starts at $+\infty$, circles the point ζ_1 in the counterclockwise sense, and returns to $+\infty$. At the beginning of the contour, $\arg(\zeta_1-x)=-\pi$, and $\arg(\zeta_2+x)=0$.

Integration of (11) by parts focuses attention on

$$[x^{-1}(d/dx)]^{l_1}x^{2l_1-2}[(d/dx)x^{-1}]^{l_2} \quad (12)$$

which, by Eq. (14) of II, is equal to

$$\sum_{\mu_1=0}^{l_1}\sum_{\mu_2=0}^{l_2}\frac{(l_1+\mu_1)!(l_2+\mu_2)!(-1)^{\mu_2}}{2^{\mu_1+\mu_2}(l_1-\mu_1)!(l_2-\mu_2)!\mu_1!\mu_2!}\left(\frac{d}{dx}\right)^{l_1-\mu_1}x^{l_1-l_2-\mu_1-\mu_2-2}\left(\frac{d}{dx}\right)^{l_2-\mu_2}, \quad (13)$$

but which also is equal to

$$x^{-1}(d/dx)^{2l_2+1}[x^{-1}(d/dx)]^{l_1-l_2-1}x^{2l_1-2l_2-2}. \quad (14)$$

If in Eq. (14), all (d/dx)'s are written to the right, the lowest power of x multiplying the derivatives is x^{-1} [i.e., $\sim x^{-1}(d/dx)^{2l_2+1}$]. We obtain the single x^{-1} term from (14), use Eq. (13) for the nonnegative powers of x, and obtain

$$\left(x^{-1}\frac{d}{dx}\right)^{l_1}x^{2l_1-2}\left(\frac{d}{dx}x^{-1}\right)^{l_2}=\frac{2^{l_2}l_2!(2l_1)!}{2^{l_1}l_1!(2l_2+1)!}x^{-1}\left(\frac{d}{dx}\right)^{2l_2+1}$$
$$+\sum_{\substack{\mu_1=0\\(l_1-l_2-2\geq\mu_1+\mu_2)}}^{l_1}\sum_{\mu_2=0}^{l_2}\frac{(l_1+\mu_1)!(l_2+\mu_2)!(-1)^{\mu_2}}{2^{\mu_1+\mu_2}(l_1-\mu_1)!(l_2-\mu_2)!\mu_1!\mu_2!}\left(\frac{d}{dx}\right)^{l_1-\mu_1}x^{l_1-l_2-\mu_1-\mu_2-2}\left(\frac{d}{dx}\right)^{l_2-\mu_2}. \quad (15)$$

Substitute Eq. (15) into Eq. (11) and use[4c]

$$x^{l_1-l_2-\mu_1-\mu_2-2}=\left(\frac{-\zeta_2}{(\zeta_1+\zeta_2)}\right)^{l_1+l_2-\mu_1-\mu_2-2}\sum_{M=0}^{l_1-l_2-\mu_1-\mu_2-2}\binom{l_1-l_2-\mu_1-\mu_2-2}{M}\left(\frac{-\zeta_1}{\zeta_2}\right)^M(\zeta_1-x)^{l_1-l_2-\mu_1-\mu_2-2-M}(\zeta_2+x)^M, \quad (16)$$

$$\int_\infty^{(\zeta_1^+)}dx(\zeta_1-x)^{l_1-l_2-n_1-\mu_2-3-M}(\zeta_2+x)^{\mu_2-n_2-1+M}$$
$$=-2i\pi(\zeta_1+\zeta_2)^{l_1-l_2-n_1-n_2-3}\Gamma(n_1+n_2-l_1+l_2+3)[\Gamma(n_1+\mu_2-l_1+l_2+3+M)\Gamma(n_2-\mu_2+1-M)]^{-1}, \quad (17)$$

and

$$\int_\infty^{(\zeta_1^+)}dx\,x^{-1}(\zeta_1-x)^{l_1-n_1-1}(\zeta_2+x)^{-l_2-n_2-2}$$
$$=-2i\pi\zeta_1{}^{l_1-n_1-1}\zeta_2{}^{-l_2-n_2-2}\Gamma(n_1+n_2-l_1+l_2+3)[\Gamma(n_1-l_1+1)\Gamma(n_2+l_2+2)]^{-1}B_{\zeta_2/(\zeta_1+\zeta_2)}(n_2+l_2+2,n_1-l_1+1), \quad (18)$$

where $B_a(b,c)$ is the incomplete beta function,[4d,6] to obtain

$$\gamma_{l_1-l_2}{}^{l_1-l_2}\{l_1>l_2\}=-\sum_{\substack{\mu_1=0\\(l_1-l_2-2\geq\mu_1+\mu_2)}}^{l_1}\sum_{\mu_2=0}^{l_2}\frac{(l_1+\mu_1)!(l_2+\mu_2)!(-1)^{\mu_2}\Gamma(n_1-\mu_1+1)}{2^{\mu_1+\mu_2}(l_1-\mu_1)!(l_2-\mu_2)!\mu_1!\mu_2!\Gamma(\mu_2-n_2)}$$
$$\times\frac{2^{l_1-l_2}(l_1-l_2)!}{(2l_1-2l_2+1)!}\Gamma(n_1+n_2+3-l_1+l_2)(\zeta_1+\zeta_2)^{\mu_1+\mu_2-n_1-n_2-1}\zeta_2{}^{l_1-l_2-\mu_1-\mu_2-2}$$
$$\times\sum_{M=0}^{(l_1-l_2-\mu_1-\mu_2-2)}\binom{l_1-l_2-\mu_1-\mu_2-2}{M}\zeta_1{}^M\zeta_2{}^{-M}\frac{\Gamma(\mu_2-n_2+M)}{\Gamma(n_1+\mu_2+3-l_1+l_2+M)}$$
$$+\frac{(l_1-l_2)!l_2!(2l_1)!}{(2l_1-2l_2+1)!l_1!(2l_2+1)!}\Gamma(n_1+n_2+3-l_1+l_2)\zeta_1{}^{l_1-n_1-1}\zeta_2{}^{-l_2-n_2-2}B_{\zeta_2/(\zeta_1+\zeta_2)}(n_2+l_2+2,n_1-l_1+1). \quad (19)$$

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[6] *Higher Transcendental Functions*, A. Erdélyi *et al.*, Eds. (McGraw-Hill Book Co., New York, 1953), Vol. 1, p. 115, Eq. (5).

Combining Eqs. (1)–(19) with Eq. (32) of II, we obtain

$$\gamma^{\lambda}{}_{n_1l_1\zeta_1;n_2l_2\zeta_2}(R)=\tfrac{1}{2}(-1)^{l_2}R^{n_1+n_2+3}$$
$$\times\sum_{\mu_1=0}^{l_1}\sum_{\mu_2=0}^{l_2}\sum_{N=0}^{\infty}(-R)^N\frac{(l_1+\mu_1)!(l_2+\mu_2)!}{2^{\mu_1+\mu_2}(l_1-\mu_1)!(l_2-\mu_2)!\mu_1!\mu_2!}\frac{2^{\lambda}\Gamma[\frac{1}{2}(n_1+n_2+3+N+\lambda)+1]}{\Gamma[\frac{1}{2}(n_1+n_2+3+N-\lambda)+1]\Gamma(n_1+n_2+5+\lambda+N)}$$
$$\times\sum_{M=0}^{N}\left(1+(-1)^{l_2+N-M}\frac{\sin\pi n_1}{\sin\pi(n_1+n_2)}+(-1)^{l_1+M}\frac{\sin\pi n_2}{\sin\pi(n_1+n_2)}\right)$$
$$\times\zeta_1{}^{M}\zeta_2{}^{N-M}[M!(N-M)!]^{-1}\Gamma(n_1-\mu_1+1+M)\Gamma(n_2-\mu_2+1+N-M)$$
$$-\sum_{\substack{\mu_1=0\\(2N+\lambda\geq\mu_1+\mu_2+2)}}^{l_1}\sum_{\mu_2=0}^{l_2}\sum_{N=0}^{\infty}R^{2N+\lambda}\frac{(l_1+\mu_1)!(l_2+\mu_2)!(-1)^{\mu_2}}{2^{\mu_1+\mu_2}(l_1-\mu_1)!(l_2-\mu_2)!\mu_1!\mu_2!}\frac{\Gamma(n_1-\mu_1+1)}{\Gamma(\mu_2-n_2)}\frac{2^{\lambda}(\lambda+N)!}{N!(2N+2\lambda+1)!}$$
$$\times\Gamma(n_1+n_2+3-2N-\lambda)(\zeta_1+\zeta_2)^{\mu_1+\mu_2-n_1-n_2-1}\zeta_2{}^{2N+\lambda-\mu_1-\mu_2-2}$$
$$\times\sum_{M=0}^{2N+\lambda-\mu_1-\mu_2-2}\binom{2N+\lambda-\mu_1-\mu_2-2}{M}\zeta_1{}^{M}\zeta_2{}^{-M}\frac{\Gamma(\mu_2-n_2+M)}{\Gamma(n_1+\mu_2+3-2N-\lambda+M)}+\delta_{\lambda,|l_1-l_2|}\Delta_{n_1l_1\zeta_1;n_2l_2\zeta_2}R^{|l_1-l_2|},\quad(20)$$

where for[5] $l_1=l_2=l$,

$$\Delta_{n_1l\zeta_1;n_2l\zeta_2}=\Gamma(n_1+n_2+3)(2l+1)^{-1}$$
$$\times[\zeta_1{}^{l-n_1-1}\zeta_2{}^{-l-n_2-2}B_{\zeta_2/(\zeta_1+\zeta_2)}(n_2+l+2,\,n_1-l+1)+\zeta_1{}^{-l-n_1-2}\zeta_2{}^{l-n_2-1}B_{\zeta_1/(\zeta_1+\zeta_2)}(n_1+l+2,\,n_2-l+1)],\quad(21)$$

for $l_1>l_2$ [Eq. (19)],

$$\Delta_{n_1l_1\zeta_1;n_2l_2\zeta_2}=(l_1-l_2)!l_2!(2l_1)![(2l_1-2l_2+1)!l_1!(2l_2+1)!]^{-1}$$
$$\times\Gamma(n_1+n_2+3-l_1+l_2)\zeta_1{}^{l_1-n_1-1}\zeta_2{}^{-l_2-n_2-2}B_{\zeta_2/(\zeta_1+\zeta_2)}(n_2+l_2+2,\,n_1-l_1+1),\quad(22)$$

and for $l_1<l_2$,

$$\Delta_{n_1l_1\zeta_1;n_2l_2\zeta_2}=(-1)^{l_1+l_2}(l_2-l_1)!l_1!(2l_2)![(2l_2-2l_1+1)!l_2!(2l_1+1)!]^{-1}$$
$$\times\Gamma(n_1+n_2+3+l_1-l_2)\zeta_1{}^{-l_1-n_1-2}\zeta_2{}^{l_2-n_2-1}B_{\zeta_1/(\zeta_1+\zeta_2)}(n_1+l_1+2,\,n_2-l_2+1).\quad(23)$$

INTEGER CASE

When n_1 is an integer while n_2 is not, Eq. (20) must be modified by

$$\sum_{M=0}^{2N+\lambda-\mu_1-\mu_2-2}\rightarrow\sum_{M=\max\{0,2N+\lambda-n_1-\mu_2-2\}}^{2N+\lambda-\mu_1-\mu_2-2}.\quad(24)$$

When n_2 is an integer while n_1 is not, the modification is

$$\sum_{M=0}^{2N+\lambda-\mu_1-\mu_2-2}\rightarrow\sum_{M=0}^{\min\{2N+\lambda-\mu_1-\mu_2-2,n_2-\mu_2\}}.\quad(25)$$

When both n_1 and n_2 are integers, Eq. (20) takes the following form [cf. Eq. (37) of II]:

$\gamma^{\lambda}{}_{n_1l_1\zeta_1;n_2l_2\zeta_2}(R)$ (n_1 and n_2 both integers)

$$=-\sum_{\substack{\mu_1=0\\(\mu_1+\mu_2\leq 2N+\lambda-2\leq n_1+n_2)}}^{l_1}\sum_{\mu_2=0}^{l_2}\sum_{N\geq 0}R^{2N+\lambda}\frac{(l_1+\mu_1)!(l_2+\mu_2)!(-1)^{\mu_2}(n_1-\mu_1)!(n_2-\mu_2)!}{2^{\mu_1+\mu_2}(l_1-\mu_1)!(l_2-\mu_2)!\mu_1!\mu_2!}\frac{2^{\lambda}(\lambda+N)!(n_1+n_2+2-2N-\lambda)!}{N!(2N+2\lambda+1)!}$$
$$\times(\zeta_1+\zeta_2)^{\mu_1+\mu_2-n_1-n_2-1}\zeta^{2N+\lambda-\mu_1-\mu_2-2}\sum_{M=\max\{0,2N+\lambda-n_1-\mu_2-2\}}^{\min\{2N+\lambda-\mu_1-\mu_2-2,n_2-\mu_2\}}\binom{2N+\lambda-\mu_1-\mu_2-2}{M}(-\zeta_1)^{M}\zeta_2{}^{-M}$$
$$\times[(n_1+\mu_2+2-2N-\lambda+M)!(n_2-\mu_2+1-M)!]^{-1}+\delta_{\lambda,|l_1-l_2|}\Delta_{n_1l_1\zeta_1;n_2l_2\zeta_2}R^{|l_1-l_2|}+\tfrac{1}{2}(-1)^{l_2}R^{n_1+n_2+3}$$
$$\times\sum_{\mu_1=0}^{l_1}\sum_{\mu_2=0}^{l_2}\sum_{N=0}^{\infty}(-R)^N\frac{(l_1+\mu_1)!(l_2+\mu_2)!}{2^{\mu_1+\mu_2}(l_1-\mu_1)!(l_2-\mu_2)!\mu_1!\mu_2!}\frac{2^{\lambda}\Gamma[\frac{1}{2}(n_1+n_2+3+N+\lambda)+1]}{\Gamma[\frac{1}{2}(n_1+n_2+3+N-\lambda)+1](n_1+n_2+4+\lambda+N)!}$$
$$\times\sum_{M=0}^{N}[1+(-1)^{l_1+n_1+M}]\zeta_1{}^{M}\zeta_2{}^{N-M}[M!(N-M)!]^{-1}(n_1-\mu_1+M)!(n_2-\mu_2+N-M)!-(-1)^{n_1+l_1+l_2}$$
$$\times\sum_{\substack{\mu_1=0\\(2N+\lambda\geq n_1+n_2+3)}}^{l_1}\sum_{\mu_2=0}^{l_2}\sum_{N}^{\infty}R^{2N+\lambda}\frac{(l_1+\mu_1)!(l_2+\mu_2)!(n_2-\mu_2)!}{2^{\mu_1+\mu_2}(l_1-\mu_1)!(l_2-\mu_2)!\mu_1!\mu_2!}\frac{2^{\lambda}(\lambda+N)!(2N+\lambda-\mu_1-\mu_2-2)!}{N!(2N+2\lambda+1)!(2N+\lambda-n_1-n_2-3)!}$$
$$\times(\zeta_1+\zeta_2)^{\mu_1+\mu_2-n_1-n_2-1}\zeta_1{}^{2N+\lambda-n_1-\mu_1-2}\zeta_2{}^{n_1-\mu_1}\sum_{M=0}^{n_1-\mu_1}\binom{n_1-\mu_1}{M}\zeta_1{}^{M}\zeta_2{}^{-M}\frac{(2N+\lambda-n_1-n_2-3+M)!}{(2N+\lambda-n_1-\mu_2-2+M)!}.\quad(26)$$

LOGARITHMIC CASE

As for the overlap integral [Eq. (41) of II], when n_1 and n_2 are not integers but n_1+n_2 is, the series [Eq. (20)] fails. The correct series is found (by L'Hospital's rule) by multiplying Eq. (20) by $\pi^{-1}(-1)^{n_1+n_2}\sin\pi(n_1+n_2)$, next taking (d/dn_2), and then setting $n_1+n_2=$ integer. Logarithmic terms enter from $(d/dn_2)R^{n_1+n_2+3+N}$, and $\psi(x)$ denotes $(d/dx)\log\Gamma(x)$. The result is

$$\gamma^{\lambda}{}_{n_1l_1\zeta_1;n_2l_2\zeta_2}(R)\ \text{(logarithmic case)}$$

$$=-\sum_{\substack{\mu_1=0\\(\mu_1+\mu_2\le 2N+\lambda-2\le n_1+n_2)}}^{l_1}\sum_{\mu_2=0}^{l_2}\sum_{N\ge 0}R^{2N+\lambda}\frac{(l_1+\mu_1)!(l_2+\mu_2)!(-1)^{\mu_2}\Gamma(n_1-\mu_1+1)}{2^{\mu_1+\mu_2}(l_1-\mu_1)!(l_2-\mu_2)!\mu_1!\mu_2!\Gamma(\mu_2-n_2)}\frac{2^{\lambda}(\lambda+N)!}{N!(2N+2\lambda+1)!}$$

$$\times(n_1+n_2+2-2N-\lambda)!(\zeta_1+\zeta_2)^{\mu_1+\mu_2-n_1-n_2-1}\zeta_2^{2N+\lambda-\mu_1-\mu_2-2}\sum_{M=0}^{2N+\lambda-\mu_1-\mu_2-2}\binom{2N+\lambda-\mu_1-\mu_2-2}{M}\zeta_1^{M}\zeta_2^{-M}$$

$$\times\frac{\Gamma(\mu_2-n_2+M)}{\Gamma(n_1+\mu_2+3-2N-\lambda+M)}+\delta_{\lambda,|l_1-l_2|}\Delta_{n_1l_1\zeta_1;n_2l_2\zeta_2}R^{|l_1-l_2|}+\tfrac{1}{2}(-1)^{l_2}R^{n_1+n_2+3}$$

$$\times\sum_{\mu_1=0}^{l_1}\sum_{\mu_2=0}^{l_2}\sum_{N=0}^{\infty}(-R)^{N}\frac{(l_1+\mu_1)!(l_2+\mu_2)!}{2^{\mu_1+\mu_2}(l_1-\mu_1)!(l_2-\mu_2)!\mu_1!\mu_2!}\frac{2^{\lambda}\Gamma[\frac{1}{2}(n_1+n_2+3+N+\lambda)+1]}{\Gamma[\frac{1}{2}(n_1+n_2+3+N-\lambda)+1]}[(n_1+n_2+4+N+\lambda)!]^{-1}$$

$$\times\sum_{M=0}^{N}\zeta_1^{M}\zeta_2^{-M}[M!(N-M)!]^{-1}\Gamma(n_1-\mu_1+1+M)\Gamma(n_2-\mu_2+1+N-M)$$

$$\times(1+(-1)^{l_1+M}\cos\pi n_1+\pi^{-1}(-1)^{M}\sin\pi n_1[(-1)^{l_2+n_1+n_2+N}-(-1)^{l_1}]\{\log R+\tfrac{1}{2}\psi[\tfrac{1}{2}(n_1+n_2+3+N+\lambda)+1]$$

$$-\tfrac{1}{2}\psi[\tfrac{1}{2}(n_1+n_2+3+N-\lambda)+1]-\psi[n_1+n_2+5+N+\lambda]+\psi(n_2-\mu_2+1+N-M)\})+(-1)^{l_1+l_2}$$

$$\times\sum_{\substack{\mu_1=0\\(2N+\lambda\ge n_1+n_2+3)}}^{l_1}\sum_{\mu_2=0}^{l_2}\sum_{N}^{\infty}R^{2N+\lambda}\frac{(l_1+\mu_1)!(l_2+\mu_2)!\Gamma(n_1-\mu_1+1)\Gamma(n_2-\mu_2+1)2^{\lambda}(\lambda+N)!}{2^{\mu_1+\mu_2}(l_1-\mu_1)!(l_2-\mu_2)!\mu_1!\mu_2!N!(2N+2\lambda+1)!}$$

$$\times[(2N+\lambda-n_1-n_2-3)!]^{-1}(\zeta_1+\zeta_2)^{\mu_1+\mu_2-n_1-n_2-1}\zeta_2^{2N+\lambda-\mu_1-\mu_2-2}$$

$$\times\sum_{M=0}^{2N+\lambda-\mu_1-\mu_2-2}\binom{2N+\lambda-\mu_1-\mu_2-2}{M}\zeta_1^{M}\zeta_2^{-M}\Gamma(\mu_2-n_2+M)[\Gamma(n_1+\mu_2+3-2N-\lambda+M)]^{-1}\pi^{-1}\sin\pi n_1$$

$$\times[-\log(\zeta_1+\zeta_2)+\psi(\mu_2-n_2)+\psi(2N+\lambda-n_1-n_2-2)-\psi(\mu_2-n_2+M)].\tag{27}$$

REMARKS

The convergence of the $\mathcal{S}^{\lambda}(R)$ series and Eq. (8) indicate that the $\gamma^{\lambda}(R)$ series converge absolutely for $|R|<\infty$.

Alternative forms of the coefficients $\gamma_N{}^{\lambda}$ may be found by expressing the $\sum_M$ of Eqs. (20), (26), and (27) in terms of hypergeometric functions[4e] and then using the linear transformation formulas.[4f] The hypergeometric-function contiguous relations[4g] also yield recurrence formulas for the $\gamma_N{}^{\lambda}$.