Expansion of a function about a displaced center

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A simpler, more general version of a formula given by Sharma for the expansion of functions of the form,
\[ F(\mathbf{r}) = f(\mathbf{r}) Y_L^m(\theta, \phi), \]
in terms of spherical harmonics and radial functions at a new origin, is derived from an earlier treatment of the expansion problem based on Fourier transforms.\textsuperscript{5}

Recently Sharma\textsuperscript{1} re-examined the problem of expressing a function of the form \[ F(\mathbf{r} - \mathbf{R}) = \sum_{l=0}^{n} \sum_{m=-l}^{l} v_{l2l}(r, R) Y_l^m(\theta, \phi) \]
in terms of spherical harmonics and radial functions at a new origin. He obtained a formula for the new radial function as a quadrature involving the old radial function. We wish to give here a simpler and more general formula, based on our earlier treatment of the expansion problem using Fourier transforms.\textsuperscript{5}

The formula we wish to establish is, in the notation of Ref. 2,
\[ v_{l2l}(r, R) = \sum_{l=0}^{n} \sum_{m=-l}^{l} C_{l2l} \frac{r}{R} Y_l^m(\theta, \phi) Y_l^m(\theta, \phi), \]

This formula is more general than Sharma’s [Eqs. (1), (16a), and (17a)-(17d) of Ref. 1] in that the vector from the old origin to the new one, \( \overline{R} \), need not lie along the \( z \) axis. At the same time it is simpler, in that the use of the well-known quantity \( C_{l2l} \) of Eq. (2)—formulas for which appear in many standard references\textsuperscript{4,5}—replaces three summations in Sharma’s Eq. (17b).

The derivation of Eqs. (1)-(4) is given, save for one final step, in Eqs. (20), (21), (17), and (18) of our earlier paper,\textsuperscript{5} which gives Eqs. (1) and (2), but instead of Eqs. (3) and (4), the pair of equations
\[ v_{l2l}(r, R) = \frac{r}{R} \int _{-\infty}^{\infty} dk k^2 j_l(kR) j_l(kR) \bar{f}(k) \]
\[ \bar{f}(k) = 4\pi l^2 \int _{0}^{\infty} dr r^{l+2} j_l(kr) \bar{f}(kr). \]

Here, \( j_l(z) \) is the usual spherical Bessel function. Substitution of Eq. (6) into Eq. (5) and interchange of the two integrations,
\[ v_{l2l}(r, R) = \int _{0}^{\infty} dr \frac{r^{l+2}}{R^2} j_l(kr) \bar{f}(kr) \int _{-\infty}^{\infty} dk k^2 j_l(kR) j_l(kR) \bar{f}(kr), \]

leads to an integral over a triple product of spherical Bessel functions that is closely related to one that occurs in the derivation of the bipolar expansion formula\textsuperscript{4} for \( r R^2 \), and that can be similarly evaluated by contour integration. The result,
\[ \int _{-\infty}^{\infty} dk k^2 j_l(kR) j_l(kr) \bar{f}(kr) \]
\[ = \frac{1}{2} \frac{(l + 2)}{l + 2} \pi R^2 \int _{0}^{\infty} dr \frac{r^{l+2}}{R^2} j_l(kr) \bar{f}(kr), \]
\[ = 0, \quad r' < |r - R| \text{ or } r' > r + R, \]
when substituted into Eq. (7), yields Eqs. (3) and (4).
3J. A. Gaunt, Philos. Trans. R. Soc. Lond. A 228, 151 (1929).