# Observations on the summability of confluent hypergeometric functions and on semiclassical quantum mechanics

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Asymptotic expansions for Airy functions and more generally confluent hypergeometric functions, which are of fundamental importance in semiclassical quantum mechanics, are summable. The Stokes lines of the expansions are cuts of the Borel sums of the power series occurring in the expansions. At a Stokes line on which the function is continuous, the asymptotic expansions change discontinuously, but their composite sums do not—a fact that greatly clarifies the role of the Stokes line. On a Stokes line itself, it is still possible to evaluate the asymptotic expansion by Borel summation via analytic continuation, and as a consequence complex expansions may have real sums, and vice versa. This observation has important implications for the significance and use of asymptotic expansions recently derived for the resonances of the LoSurdo-Stark effect and for the energy eigenvalues of $H_2^+$. For both of these problems the physical values of the expansion parameters, the electric field strength and the reciprocal of the internuclear distance, lie on Stokes lines.

## I. INTRODUCTION

Semiclassical methods recently have led to high-order asymptotic expansions, including exponentially small subdominant terms, for two real physical problems that are themselves important as prototypes: the LoSurdo-Stark effect in hydrogen[1,2] and the hydrogen molecular ion $H_2^+$.[3–7] In the case of $H_2^+$, the double-well gap enters only through the exponentially small terms, which are clearly essential. However, the real values of the electric field strength and the reciprocal internuclear distance, which are the expansion parameters for these two problems, fall on Stokes lines of the expansions, i.e., the lines on which the coefficients of the subdominant expansions change discontinuously. Consequently, ambiguity might seem to arise about the meaning of the expansions so obtained. Elimination of this ambiguity is what has motivated our reexamination of the asymptotic expansions of the confluent hypergeometric functions that underlie the semiclassical approach.

Despite widespread computational usefulness, asymptotic expansions (by which we mean not just asymptotic power series, but asymptotic expansions as defined in Refs. 8 and 9, and which are exemplified by the usual asymptotic expansions for the confluent hypergeometric functions) are often thought to be of limited numerical accuracy (ultimately they diverge) and to be characterized by nonunique parentage (different functions can have the same asymptotic expansion). Confusion or, at best, an apologetic awkwardness about Stokes lines where the "coefficient" of a subdominant series changes discontinuously, is manifest in many textbook discussions of the "meaning" or "interpretation" of asymptotic expansions near Stokes lines.[8,9] In practical applications of solving differential equations by semiclassical and Jeffreys-Wentzel-Kramers-Brillouin (JWKB) methods, ambiguity in the coefficient of a subdominant expansion can be particularly vexing, especially when the physical domain of the expansion variable is a Stokes line of the expansion. It does not seem to be widely appreciated that by using the notion of Borel summability, these deficiencies and ambiguities disappear (i) for asymptotic expansions of confluent hypergeometric functions, (ii) for those physical problems whose perturbation series are Borel summable, and perhaps to a large extent (iii) for those physical problems that are based on the confluent hypergeometric expansions, the most important examples being the semiclassical and quasisemiclassical solutions of the Schrödinger equation.

For example, since the divergent Rayleigh-Schrödinger perturbation theory (RSPT) for the LoSurdo-Stark effect in hydrogen is Borel summable,[10] if in a practical calculation[11] the number of terms is increased, the accuracy of the Borel sum will increase. Moreover, both the real *and imaginary* parts of the resonance eigenvalues are obtained[11] by the Borel sum of *only* the RSPT series, whose coefficients are real, while the subdominant expansion[1,2] for the imaginary part, which can be explicitly and independently computed, must be totally ignored when calculating the resonance energies by Borel summing the RSPT—a Stokes-line situation that will be clarified further below.

The divergent asymptotic expansion for the energy of the hydrogen molecular ion $H_2^+$ with respect to internuclear distance is a second more complicated example.[3–7] In addition to the RSPT series and the exponentially small gap series mentioned above, there exists a sequence of successively more subdominant series,[5–7] some of which are *imaginary*. Since the energy is real, imaginary terms might seem suspect. As in the case of the LoSurdo-Stark effect, the RSPT series itself has a Borel sum that is complex. The first imaginary subdominant series for $H_2^+$ consists of the explicit counterterms to cancel the implicit imaginary part of the Borel sum of the RSPT series. In fact, it was the unexpectedness of this phenomenon that was the immediate stimulus for the present study.

The main purpose of this paper is to call attention to the Borel summability of the power series appearing in the confluent hypergeometric asymptotic expansions and the implications for semiclassical quantum mechanics. The specific points are the following. (i) The *appropriate* asymptotic expansions for the confluent hypergeometric functions are summable to the confluent hypergeometric functions. *By this we mean that the power series appearing in the expansions are Borel summable.* (In fact, one would guess that these asymptotic expansions may have been a contributing factor to Borel's invention of Borel summability, but nevertheless it is certainly true that their summability is no longer widely known nor the implications on the role of the subdominant expansions widely understood.) (ii) The cuts of the Borel sums of the power series are the Stokes lines of the asymptotic expansions. This is an observation whose significance is probably deeper than is immediately apparent. (iii) The "values" of the expansions on the Stokes lines are obtained from the sums of the two different expansions, each on its appropriate side, by analytic continuation. When the Stokes line is not a cut of the confluent hypergeometric function, the values so obtained must be identical. That the expansions change discontinuously at a Stokes line is for this reason absolutely necessary for the sums on either side to coincide at the Stokes line, since the Stokes line is a cut of the Borel sum of the power series of the dominant expansion on each side. This point is a powerful key to clarifying one of the most confusing characteristics of asymptotic expansions, the discontinuous change in form at Stokes lines, and it will be illustrated particularly clearly by the Airy Bi($z$) function, whose "standard expansion" for $z$ near the positive real axis is *not* summable to Bi($z$), and whose summable expansion is a simple example of the explicitly-complex-expansion-representing-a-real-function phenomenon that we first encountered[5–7] with $H_2^+$, and that had been seen in part by Zinn-Justin[12,13] in double-well oscillators. (iv) In applications such as the JWKB method, if the underlying asymptotic expansions of the Airy and confluent hypergeometric functions are taken in the sense of the summability defined above, then the presence of subdominant expansions is automatically unambiguous. This is particularly important in double-well problems, such as $H_2^+$ mentioned above, for which the physics takes place on a Stokes line, and for which there is a hierarchy of subdominant series contributing to the complete asymptotic expansion for the energy. Similarly, in quantum-defect theory[14] the notion that when approaching a Stokes line a real function *should* have a complex asymptotic expansion, simply undoes a "persistent paradox in the QDT literature."[14]

## II. SUMMABILITY OF AIRY EXPANSIONS

We focus first on the Airy functions for concreteness and because of their importance in the JWKB method and in Langer's method[15] for solution of differential equations by uniform asymptotic expansions. Both of the Airy functions, Ai($z$) and Bi($z$), are entire and are special cases of the confluent hypergeometric function, the more general versions of which are treated later. Two asymptotic expansions are given for Ai($z$) in the *Handbook of Mathematical Functions*[16] [Eqs. (10.4.59) and (10.4.60)]:

$$\mathrm{Ai}(z) \sim \tfrac{1}{2}\pi^{-1/2} z^{-1/4} e^{-\zeta} \sum_0^\infty (-1)^k c_k \zeta^{-k}\,, \tag{1}$$

$$\mathrm{Ai}(-z) \sim \pi^{-1/2} z^{-1/4} \times \left[ \sin(\zeta+\tfrac{1}{4}\pi) \sum_0^\infty (-1)^k c_{2k} \zeta^{-2k} - \cos(\zeta+\tfrac{1}{4}\pi) \sum_0^\infty (-1)^k c_{2k+1} \zeta^{-2k-1} \right], \tag{2}$$

$$= \tfrac{1}{2}\pi^{-1/2} z^{-1/4} e^{i(\zeta-\pi/4)} \sum_0^\infty (-i)^k c_k \zeta^{-k} + \tfrac{1}{2}\pi^{-1/2} z^{-1/4} e^{-i(\zeta-\pi/4)} \sum_0^\infty i^k c_k \zeta^{-k}\,, \tag{3}$$

where

$$\zeta = \tfrac{2}{3} z^{3/2}, \quad c_k = \frac{\Gamma(k+\frac{5}{6})\Gamma(k+\frac{1}{6})}{\Gamma(\frac{5}{6})\Gamma(\frac{1}{6}) 2^k k!}\,. \tag{4}$$

Equation (3) is a more transparent exponential version of Eq. (2). It emphasizes that there is really only one *fundamental* Airy function expansion, Eq. (1): the others have $z$ replaced by $ze^{i\theta}$ for suitable $\theta$. *In practice it is necessary to establish summability in the above sense for only the fundamental expansion. Summability of the other expansions are simple consequences of the summability of the fundamental expansion.* The domains of applicability given in Ref. 16 are $|\arg(z)| < \pi$ for Eq. (1) and $|\arg(z)| < \frac{2}{3}\pi$ for Eqs. (2) and (3). These domains, if one notes that Eqs. (2) and (3) refer to "$-z$," overlap. The domains for summability are smaller and do not overlap (vide infra).

The summability of the fundamental expansion (1) can be seen by elementary means. It is also possible to establish Borel summability of the power series of Eq. (1) by showing that $z^{1/4} e^{\zeta} \mathrm{Ai}(z)$ satisfies the conditions for the Watson-Nevanlinna theorem,[17] but the direct, elementary demonstration is more instructive. Borel summation has three steps. (i) The $k$th term is multiplied by $t^k k!$, and the "Borel transform" is summed. In this case the result for the power-series component of Eq. (1), $z^{1/4} e^{\zeta} \mathrm{Ai}(z)$, is

$$B(z,t) = \tfrac{1}{2}\pi^{-1/2} \sum_0^\infty (-1)^k c_k (t/\zeta)^k / k! \qquad (|t/\zeta| < 2)\,, \tag{5}$$

$$= \tfrac{1}{2}\pi^{-1/2} {}_2F_1(\tfrac{1}{6}, \tfrac{5}{6}; 1; -\tfrac{1}{2} t/\zeta)\,. \tag{6}$$

(ii) The Borel transform $B(z,t)$ is analytically continued to all of $0 \le t < \infty$. The Gauss hypergeometric function[16] automatically provides the continuation, so long as $\zeta$ is not real and negative; i.e., $|\arg(\zeta)| < \pi$, which is equivalent to $|\arg(z)| < \frac{2}{3}\pi$. (iii) The $B(z,t)$ is multiplied by $e^{-t}$ and integrated from zero to infinity:

$$\int_0^\infty dt\, e^{-t} B(z,t) = \tfrac{1}{2}\pi^{-1/2}\int_0^\infty dt\, e^{-t}\,{}_2F_1(\tfrac{1}{6},\tfrac{5}{6};1;-\tfrac{1}{2}t/\zeta)\ . \quad (7)$$

The integral (7) can be "looked up" in standard references, but with caution: it is a special case of an integral that is given correctly [Eq. (30), p. 78] in Buchholz[18] and correctly once in Erdélyi *et al.*,[19] but that is given incorrectly twice in Erdélyi *et al.*,[19] twice in Gradshteyn and Ryzhik,[20] and twice in Roberts and Kaufman.[21] The general integral is

$$\int_0^\infty dt\, e^{-zt} t^{a-1}\,{}_2F_1(\tfrac{1}{2}+\tfrac{1}{2}m-b,\tfrac{1}{2}-\tfrac{1}{2}m-b;a;-t) = z^{-a-b}e^{z/2}\Gamma(a)W_{b,m/2}(z)\ . \quad (8)$$

Here $W_{b,m/2}(z)$ is the usual Whittaker confluent hypergeometric function,[16,18] Ai$(z)$ is just that special case of the Whittaker $W$ function[22] for which

$$\mathrm{Ai}(z)=\tfrac{1}{2}\pi^{-1/2}z^{1/4}e^{-\zeta}\int_0^\infty dt\, e^{-t}\,{}_2F_1(\tfrac{1}{6},\tfrac{5}{6};1;-\tfrac{1}{2}t/\zeta) \quad [\,|\arg(z)|<\tfrac{2}{3}\pi]\ . \quad (9)$$

In this direct way, one verifies the summability of the expansion (1) to Ai$(z)$. Let us look more carefully at the domain. If $\arg(z)=\pm\frac{2}{3}\pi$, then $\arg(\zeta)=\pm\pi$, and the branch point of the hypergeometric function at $t=-2\zeta$ lies on the positive real $t$ axis, the integration path in Eqs. (7) and (9). Consequently, the Borel sum defined by the integral has a cut on the rays $\arg(z)=\pm\frac{2}{3}\pi$, which are also recognizable as the Stokes lines[8,9] of the Ai$(z)$ expansion. Of course the Borel sums can be analytically continued beyond $\arg(z)=\pm\frac{2}{3}\pi$ simply by changing the integration path in Eqs. (7) and (9) to a suitable ray on which $\arg(-t/\zeta)\neq 0$, which, in fact, is how one could calculate Ai$(z)$ via Borel summation in a neighborhood of the Stokes line. Note, however, that without analytic continuation, the standard domain of Borel summability is $|\arg(z)|<\frac{2}{3}\pi$ [not $|\arg(z)|<\pi$].

The summability of the series (3) [and (2)] follows with relatively little effort from the summability of (1) and a well-known identity [Eq. (10.4.7) of Ref. 16]:

$$\mathrm{Ai}(-z)=e^{\pi i/3}\mathrm{Ai}(ze^{\pi i/3})+e^{-\pi i/3}\mathrm{Ai}(ze^{-\pi i/3})\ . \quad (10)$$

Equation (1) can be used for both of the Ai$(ze^{\pm\pi i/3})$ provided that $|\arg(ze^{\pm\pi i/3})|<\frac{2}{3}\pi$ for both signs simultaneously, namely, when $|\arg(z)|<\pi/3$, and the result is exactly Eq. (3). That is, the summability of Eq. (3) [and (2)] for $|\arg(z)|<\pi/3$ [not $|\arg(z)|<\frac{2}{3}\pi$] follows from the identity (10) and the summability of Eq. (1). Notice that the domains for summability of Eqs. (1) and (3) are not overlapping and cover the full plane [when one takes account of the $-z$ in Eq. (3)], except for the Stokes lines, which in turn can be included by analytic continuation from either side.

The summable Bi$(z)$ asymptotic expansions follow from the summability of the fundamental expansion (1) and a well-known identity [Eq. (10.4.9) of Ref. 16]:

$$\mathrm{Bi}(z)=\mp i\,\mathrm{Ai}(z)+2e^{\pm\pi i/6}\mathrm{Ai}(ze^{\pm 2\pi i/3})\ . \quad (11)$$

A second identity that combines Eq. (11) with (10) gives Bi$(-z)$:

$$\mathrm{Bi}(-z)=e^{-\pi i/6}\mathrm{Ai}(ze^{\pi i/3})+e^{\pi i/6}\mathrm{Ai}(ze^{-\pi i/3})\ . \quad (12)$$

To use Eq. (1) in Eq. (11), both $|\arg(z)|<\frac{2}{3}\pi$ and one of $|\arg(ze^{\pm 2\pi i/3})|<\frac{2}{3}\pi$ must hold simultaneously. There are two cases: the upper sign with $-\frac{2}{3}\pi<\arg(z)<0$; and the lower sign with $0<\arg(z)<\frac{2}{3}\pi$. Use of Eq. (1) in Eq. (12) is completely analogous with its use in Eq. (10), the common domain for both expansions being $|\arg(z)|<\pi/3$. The resulting three equations are

$$\mathrm{Bi}(z)\sim \pi^{-1/2}z^{-1/4}e^{\zeta}\sum_0^\infty c_k\zeta^{-k} - i\tfrac{1}{2}\pi^{-1/2}z^{-1/4}e^{-\zeta}\sum_0^\infty(-1)^k c_k\zeta^{-k} \quad [-\tfrac{2}{3}\pi<\arg(z)<0]\ , \quad (13)$$

$$\mathrm{Bi}(z)\sim \pi^{-1/2}z^{-1/4}e^{\zeta}\sum_0^\infty c_k\zeta^{-k} + i\tfrac{1}{2}\pi^{-1/2}z^{-1/4}e^{-\zeta}\sum_0^\infty(-1)^k c_k\zeta^{-k} \quad [0<\arg(z)<\tfrac{2}{3}\pi]\ , \quad (14)$$

$$\mathrm{Bi}(-z)\sim \tfrac{1}{2}\pi^{-1/2}z^{-1/4}e^{i(\zeta+\pi/4)}\sum_0^\infty(-i)^k c_k\zeta^{-k} + \tfrac{1}{2}\pi^{-1/2}z^{-1/4}e^{-i(\zeta+\pi/4)}\sum_0^\infty i^k c_k\zeta^{-k} \quad [\,|\arg(z)|<\pi/3]\ . \quad (15)$$

Equations (13) and (14) for Bi$(z)$, whose sums may be analytically continued to $\arg(z)=0$, are examples of a *complex expansion with a real sum.* Contrast these expansions with those given in Abramowitz and Stegun.[16] Equations (13) and (14) correspond to Eq. (10.4.65) (after some minor manipulation), while Eq. (15) matches Eq. (10.4.64), except that the domains here stop at the Stokes lines. The standard expansion for Bi$(z)$ $(z>0)$, Eq. (10.4.63) of Ref. 16,

$$\mathrm{Bi}(z)\sim\pi^{-1/2}z^{-1/4}e^{\zeta}\sum_0^\infty c_k\zeta^{-k} \quad (16)$$

[N.B., not summable to Bi$(z)$], is not obtained here at all. Comparison of Eq. (16) with Eqs. (11), (13), and (14) shows that the sum of the standard expansion (16) is Bi$(z)\pm i$ Ai$(z)$, the sign depending whether $0<\mp\arg(z)<\frac{2}{3}\pi$. Intuition would suggest that a real function should have a real asymptotic expansion. That is not the case here because the real axis is a cut of the Borel sum of the power series of the expansion (16). The sum of the "real" expansion approaches complex values from either side of the real axis, and an explicitly complex expansion [Eqs. (13) and (14)] is required for the implicit and explicit imaginary parts to cancel and leave a real numerical result. This is exactly the phenomenon encountered in $H_2^+$ by us[5–7] and in the double-well oscillator by Zinn-Justin.[12,13]

This same phenomenon has an analog in the resonances of the LoSurdo-Stark effect in hydrogen, which is an example of a real expansion with a complex sum. There the energy levels are complex resonances. Intuitively one would expect the complex energy to have a complex asymptotic expansion. The Borel sum of the RSPT energy series has a cut on the real axis and already carries the correct imaginary part,[11] although only by analytic continuation. The explicit series[1,2] derived for the imaginary part is the direct asymptotic expansion for the imaginary part of the resonance; it can also be understood as one-half the discontinuity in the Borel sum on the real axis. One can calculate the imaginary part by either expansion, but both series do not contribute simultaneously to the same summable expansion.

## III. SUMMABILITY OF THE CONFLUENT HYPERGEOMETRIC EXPANSIONS

As has already been remarked, the Airy function is a special case of the confluent hypergeometric function, Eqs. (8) and (9). The confluent hypergeometric functions come in two slightly different versions:[16,18] Kummer and Whittaker. We discuss Whittaker's functions, which are more closely related to the hydrogenic eigenfunctions, but all results for Whittaker functions hold, *mutatis mutandis*, for the Kummer functions. Of the two Whittaker functions, $M_{b,m/2}(z)$ and $W_{b,m/2}(z)$, the asymptotic expansion [Eq. (1a), p. 90 of Ref. 18] for $W_{b,m/2}(z)$ is the *fundamental* one in the same sense as the Ai$(z)$ expansion (1):

$$W_{b,m/2}(z) \sim z^b e^{-z/2} \sum_0^\infty (-1)^k \frac{\Gamma(\frac{1}{2}+\frac{1}{2}m-b+k)\Gamma(\frac{1}{2}-\frac{1}{2}m-b+k)}{\Gamma(\frac{1}{2}+\frac{1}{2}m-b)\Gamma(\frac{1}{2}-\frac{1}{2}m-b)} z^{-k}/k!\,, \tag{17}$$

$$\sim z^b e^{-z/2}\, {}_2F_0(\tfrac{1}{2}+\tfrac{1}{2}m-b, \tfrac{1}{2}-\tfrac{1}{2}m-b;;-1/z)\,, \tag{18}$$

where the ${}_2F_0$ denotes the usual generalized hypergeometric function.[16] The domain usually stated for Eq. (17), $|\arg(z)| < \frac{3}{2}\pi$, is larger than the domain for summability. One can calculate directly the Borel transform of the generalized hypergeometric series

$$B(z,t) = \sum_0^\infty (-1)^k \frac{\Gamma(\frac{1}{2}+\frac{1}{2}m-b+k)\Gamma(\frac{1}{2}-\frac{1}{2}m-b+k)}{\Gamma(\frac{1}{2}+\frac{1}{2}m-b)\Gamma(\frac{1}{2}-\frac{1}{2}m-b)} \times (t/z)^k/(k!)^2\,, \tag{19}$$

$$= {}_2F_1(\tfrac{1}{2}+\tfrac{1}{2}m-b, \tfrac{1}{2}-\tfrac{1}{2}m-b;1;-t/z)\,. \tag{20}$$

Similar to the Airy function, the circle of convergence is $|t| < |z|$, and the Gauss hypergeometric function carries the analytic continuation to all $t \geq 0$, provided that $z$ is not real and negative. The result of multiplying by $e^{-t}$ and integrating from 0 to $\infty$ is exactly given by Eq. (8) with $a=1$. That is,

$$W_{b,m/2}(z) = z^b e^{-z/2} \times \int_0^\infty dt\, e^{-t}\, {}_2F_1(\tfrac{1}{2}+\tfrac{1}{2}m-b, \tfrac{1}{2}-\tfrac{1}{2}m-b;1;-t/z) \quad [\,|\arg(z)| < \pi]\,. \tag{21}$$

Thus the power series in Eq. (17) for $W_{b,m/2}(z)$ is Borel summable in the plane cut from 0 to $-\infty$, which in this case also happens to be a branch cut for $W_{b,m/2}(z)$, as well as for the Borel sum.

The summable expansions for the Whittaker $M_{b,m/2}(z)$ follow from the summable expansion (17) for the $W_{b,m/2}(z)$ function, the definition of $M_{b,m/2}(z)$ in terms of $W$'s [Eq. (20a), p. 19 of Ref. 18], and consideration of the domains in which the power series in both $W$'s entering $M$ are simultaneously Borel summable. The results are directly analogous to Eqs. (11), (13), and (14) for Bi$(z)$:

$$M_{b,m/2}(z) = \frac{\Gamma(1+m)e^{\pm\pi i b}}{\Gamma(\frac{1}{2}+\frac{1}{2}m-b)} W_{-b,m/2}(ze^{\pm\pi i}) + \frac{\Gamma(1+m)e^{\pm\pi i(b-1/2-m/2)}}{\Gamma(\frac{1}{2}+\frac{1}{2}m+b)} W_{b,m/2}(z)\,, \tag{22}$$

$$\sim \frac{\Gamma(1+m)z^{-b}e^{z/2}}{\Gamma(\frac{1}{2}+\frac{1}{2}m-b)}\, {}_2F_0(\tfrac{1}{2}+\tfrac{1}{2}m+b, \tfrac{1}{2}-\tfrac{1}{2}m+b;;+1/z) + \frac{\Gamma(1+m)z^b e^{-z/2}}{\Gamma(\frac{1}{2}+\frac{1}{2}m+b)} e^{\pi i(b-1/2-m/2)}\, {}_2F_0(\tfrac{1}{2}+\tfrac{1}{2}m-b, \tfrac{1}{2}-\tfrac{1}{2}m-b;;-1/z) \quad [-\pi < \arg(z) < 0]\,, \tag{23}$$

$$\sim \frac{\Gamma(1+m)z^{-b}e^{z/2}}{\Gamma(\frac{1}{2}+\frac{1}{2}m-b)}\, {}_2F_0(\tfrac{1}{2}+\tfrac{1}{2}m+b, \tfrac{1}{2}-\tfrac{1}{2}m+b;;+1/z) + \frac{\Gamma(1+m)z^b e^{-z/2}}{\Gamma(\frac{1}{2}+\frac{1}{2}m+b)} e^{-\pi i(b-1/2-m/2)}\, {}_2F_0(\tfrac{1}{2}+\tfrac{1}{2}m-b, \tfrac{1}{2}-\tfrac{1}{2}m-b;;-1/z) \quad [0 < \arg(z) < \pi]\,. \tag{24}$$

The only difference between Eqs. (23) and (24) is the phase factor of the subdominant series. It is precisely this phase factor that plays the key role in deriving the imaginary series in the $H_2^+$ problem. When $z$ is real and positive, the $M$ function is real. But $z>0$ is a cut of the Borel sum of the power series ${}_2F_0(\frac{1}{2}+\frac{1}{2}m+b,\frac{1}{2}-m/2+b;;+1/z)$, and consequently the asymptotic expansion appears complex and discontinuous. To include the real axis, one may analytically continue the Borel sum from above or below. In either case the result is the same: the real, continuous value of $M_{b,m/2}(z)$.

In summary, the power series for the standard asymptotic expansion for $W_{b,m/2}(z)$ is Borel summable on the cut $z$ plane. The standard expansions for $M_{b,m/2}(z)$ are summable in the upper or lower half-plane, and the real axis can be included by analytic continuation. The cuts of the Borel sum of the power series are the Stokes lines. Despite the dominant-subdominant relationship that exists between the two component series for $M_{b,m/2}(z)$ in various regions of the plane, the ambiguity in the Poincaré sense of how to fix the contribution of the subdominant series is completely removed by the Borel sense, since both subseries are required throughout the domain for summability.

## IV. CONCLUDING REMARKS

Borel summability provides a one-to-one correspondence between a function and a divergent asymptotic power series. The confluent hypergeometric function asymptotic expansions and various examples of RSPT for hydrogen and the anharmonic oscillator[23] are summable through the Borel summability of the component power series. Stokes lines are cuts of the Borel sums. If a Stokes line is not a cut of the parent function, then the form of the summable asymptotic expansion *necessarily* changes at a Stokes line. Summable asymptotic expansions are valid on the Stokes lines themselves by analytic continuation from either side. As a consequence, the realness or complexity of the parent function near a Stokes line does not in the naive sense imply realness or complexity of the *formal* expansion near the Stokes line. Because of inherent uniqueness and the unambiguous inclusion of subdominant series, summation in the above sense forms a satisfactory conceptual framework for the solution of the Schrödinger equation by semiclassical methods.

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[22] The relationship is $\mathrm{Ai}(z)=\frac{1}{2}\pi^{-1/2}z^{-1/4}W_{0,1/3}(2\zeta)$.
[23] S. Graffi, V. Grecchi, and B. Simon, Phys. Lett. **32B**, 631 (1970).