The $1/R$ Expansion for $H^+_2$:
Analyticity, Summability, and Asymptotics

S. GRAFFI

Dipartimento di Matematica, Università di Bologna,
I – 40127 Bologna, Italy

V. GRECCHI

Dipartimento di Matematica, Università di Modena,
I – 41100 Modena, Italy

E. M. HARRELL II*

School of Mathematics, Georgia Institute of Technology,
Atlanta, Georgia 30332-0160

AND

H. J. SILVERSTONE†

Department of Chemistry, The Johns Hopkins University,
Baltimore, Maryland 21218

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It is proved that the $1/R$ expansion for $H^+_2$ is divergent and Borel summable to a complex
eigenvalue of a non-self-adjoint operator, which has the same $1/R$ expansion. The Borel sum is
related to the $H^+_2$ system as follows: its real part agrees with the eigenvalue doublet
asymptotically to all orders, and its imaginary part determines the asymptotics of the $1/R$
expansion coefficients via a dispersion relation. A rigorous estimate of the leading behavior of
the imaginary part is obtained, and as a consequence the approximate formula of Brézin and
Zinn-Justin relating the square of the eigenvalue gap to the asymptotics of the $1/R$ expansion
is put on a rigorous basis. © 1985 Academic Press, Inc.

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analyticity, and summability. IV. Imaginary parts, asymptotics, and the formula of Brézin and
Zinn-Justin. Appendix A. Appendix B. List of symbols

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I. INTRODUCTION

Consider the two-center problem of an electron in the field of two fixed point charges $Z_A, Z_B$ at a distance $R$ apart. In non-relativistic quantum mechanics its Hamiltonian is

$$H(R, Z_A, Z_B) = -\frac{1}{2} A - Z_A|x|^{-1} - Z_B|x - R\hat{e}|^{-1}$$  \hspace{1cm} (1.1)

in atomic units, with $x \in \mathbb{R}^3$, $\hat{e} = (1, 0, 0)$. If $Z_A = Z_B = 1$ this describes the hydrogen molecular ion $\text{H}_2^+$ in the clamped nuclei approximation, which is an important double-well problem having the virtue of being separable. In the normalization of (1.1) the formal limit as $R \to \infty$ is the Hamiltonian of hydrogen.

The series in negative powers of $R$ obtained by expanding $|x - R\hat{e}|^{-1}$ and applying Rayleigh–Schrödinger perturbation theory exists, is called the $1/R$ expansion, and is a classic textbook example [1]. However, (1.1) also furnishes a classic example of unstable perturbation: although the $\text{H}_2^+$ eigenvalues approach those of hydrogen as $R \to \infty$ (first proved by Aventini and Seiler [2]), and the rate of convergence is correctly described by the asymptotic $1/R$ expansion (Morgan and Simon [3]), they are doubly asymptotically degenerate as $R \to \infty$. That is, near any given bound state of $\text{H}_2^+$ for $1/R$ small enough there are two bound states of $\text{H}_2^+$ with an energy gap of order $R^{2k+1} \exp(-R/n)$, where $n$ and $k$ are the usual principal and parabolic quantum numbers [1].

The instability is a double-well phenomenon, (1.1) being somewhat analogous to the one-dimensional double-well anharmonic oscillator $p^2 + x^2(1 + gx)^2$. It is similarly clear that the $1/R$ expansion cannot be Borel summable to an eigenvalue. How could the series decide which eigenvalue to sum to? Numerically, the series has been found [3] to be factorially divergent with coefficients of one sign, in analogy to the double-well oscillator [4].

In addition, it has been discovered by Brézin and Zinn-Justin [5], also numerically, that the square of the gap between the eigenvalue doublet converging to the hydrogen ground state is related to the asymptotics of the $1/R$ expansion. This typical non-perturbative tunneling quantity, $O(R^2 \exp^{-2R})$ for the ground state, is reminiscent of the resonance width in the Lo Surdo–Stark effect, for which a one-to-one relationship with the perturbation series has been proved and exploited [6, 31]. That proof was based on the Borel summability of the perturbation series to the resonance [7]. More specifically, the imaginary part of the Borel sum determines the asymptotics of the perturbation series and, conversely, the asymptotic behavior of the series determines the leading behavior of the imaginary part of the sum. In the case of the Lo Surdo–Stark effect the Borel sum is a resonance in the standard sense of dilatation analyticity [7–10]. Although the imaginary part of the double-well oscillator eigenvalue does not seem to have a physical interpretation as a resonance, it determines the eigenvalue gap asymptotically [11].

The purpose of this paper is to show these phenomena rigorously in the case of the $1/R$ expansion of $\text{H}_2^+$. We will prove that the Borel sum of the $1/R$ expansion exists as the complex eigenvalue of a non-self-adjoint problem that has the same
1/R expansion as $H_2^+$ but is stable as $R \to \infty$. The imaginary part of the Borel sum determines the asymptotics of the perturbation coefficients and conversely. (For a general overview of this kind of result for the anharmonic oscillator and the Lo Surdo–Stark effect, see Simon [12].) Furthermore, we derive rigorously the asymptotic form of the imaginary part of the Borel sum, which verifies the approximate formula of Brézin and Zinn-Justin. Notice that the 1/R expansion not only determines the position of the $H_2^+$ doublet asymptotically, but also the gap to leading order.

Although this result is closely analogous to the ones for the double-well oscillator and the Lo Surdo–Stark effect mentioned above, it requires a more subtle analysis, looking into the relationship between $H_2^+$ and the system of an electron in the field of a stationary proton and a stationary anti-proton,

$$H'(R, Z_A, -Z_A) = -\frac{1}{2} A - Z_A |x|^{-1} + Z_A |x + R\hat{e}|^{-1}$$  \hspace{1cm} (1.2)

(in [14] $H'$ was denoted $K$) the 1/R expansion of which is identical to that of $H_2^+$ but with $R$ replaced by $-R$, so that the signs alternate. A plausible starting point of the analysis would be to prove Borel summability of eigenvalues of (1.2) and then analytically continue from $-R$ to $+R$, where they should develop a branch cut and thus an imaginary part. However, we shall see that although (1.2) is a stable, single-well problem, its alternating-sign 1/R expansion is not Borel summable to its eigenvalues, thus answering in the negative a question raised by Morgan and Simon [3]. Incidentally, we remark that this is, to our knowledge, the only example of this type which has a direct physical interest.

The identification of the Borel sum will involve relating (1.1) and (1.2) in a more subtle way, using the separability in elliptic coordinates to be implemented in Section II, which also contains a detailed description of the generation of the 1/R expansion from the separated equations. In Section III we shall describe the stability, analyticity, and implicit function arguments which, together with the remainder estimates, allow the Borel sum to be identified as a function holomorphic in some half-disk $|1/R| < M$, $\text{Im } R > 0$, which admits analytic continuation across the branch cut along the real axis (Theorem III.2). In Section IV we shall determine the leading exponential order of the imaginary part of the Borel sum (Theorem IV.1) and establish the dispersion relation connecting it to the asymptotics of the 1/R expansion. The proof of the Brézin–Zinn-Justin formula (Corollary IV.2) will then be a simple consequence of this and the known estimates of the eigenvalue gap [13]. Finally, we collect some technical lemmas on Borel summability of composed and implicit function in Appendix A and the JWKB estimates of the tunneling factors needed to estimate imaginary parts in Appendix B.

We conclude this Introduction by mentioning that this work represents the first of the two papers announced in Ref. [14], in which part of the above results are briefly described together with a semiclassical procedure for generating all exponentially small corrections to the 1/R expansion for the bound states of $H_2^+$. 
II. Separated Equations and Perturbation Theory

Let us begin by collecting some well-known relevant facts about the family of Schrödinger operators describing the general two-center problem. Since, as will become evident, the natural variable is \( \rho = 1/R \) rather than \( R \), the operator (1.1) will henceforth be denoted \( H(\rho, Z_A, Z_B) \). Unless otherwise specified, the operator-theoretic notation used throughout this paper is that of Reed and Simon [15].

**Proposition II.1.** Let \( \rho^{-1} = R > 0 \), and \( Z_A, Z_B \in \mathbb{R} \). Let \( H(\rho, Z_A, Z_B) \) denote the family of operators on \( L^2(\mathbb{R}^3) \) defined as the action of \(-\frac{1}{2} \Delta - Z_A |x|^{-1} - Z_B |x - R\hat{e}|^{-1}\) on the domain of definition \( H^2(\mathbb{R}^3) \) (Sobolev space), and let \( H_0(Z_A) \) denote the hydrogen operator, i.e., the action of \(-\frac{1}{2} \Delta - Z_A |x|^{-1}\) on the same domain. Then:

1. \( H(\rho, Z_A, Z_B) \) is self-adjoint and bounded below.
2. \( \sigma_{\text{ess}}(H(\rho, Z_A, Z_B)) = \sigma_{\text{ac}}(H(\rho, Z_A, Z_B)) = [0, +\infty) \).
3. Let \( E(\rho, Z_A, Z_B) \) be an eigenvalue of \( H(\rho, Z_A, Z_B) \). Then \( \rho \to E(\rho, \cdot) \) is continuous, and \( \lim_{\rho \to 0} E(\rho, \cdot) \) exists and is an eigenvalue of \( H_0(Z_A) \) if \( Z_A > 0 \).
4. If \( Z_A > 0, Z_B < 0 \), the eigenvalues of \( H_0(Z_A) \) are stable (in the sense of Kato [16, Sect. VIII.1.4]) for \( \rho > 0 \) small.
5. Fix \( Z_A = Z_B > 0 \), and recall that the eigenvalues of \( H_0(Z_A) \) are \(-Z_A^2/2n^2\), \( n = 1, 2, \ldots \), with multiplicities \( n^2 \). For each such unperturbed eigenvalue and any open interval \( I \) containing only that unperturbed eigenvalue, there exists \( M > 0 \) such that for \( \rho < M \) there are precisely \( 2n^2 \) eigenvalues in \( I \). The cluster of eigenvalues in \( I \) is organized in exponentially close pairs, and the two eigenvalues \( E_\pm \sim -Z_A^2/2 \) in particular satisfy

\[
\Delta E(\rho, Z_A) \equiv E_+(\rho, Z_A) - E_-(\rho, Z_A) = O(Re^{-K}).
\]

6. The Rayleigh–Schrödinger perturbation expansion in powers of \( \rho \) near \( E_0(Z_A) \) in (5) exists and represents an asymptotic expansion for both eigenvalues \( E_{\pm}(\rho, \cdot) \) as \( \rho \to 0 \).

**Remarks.** (1) For the general analysis of the operator family \( H(\rho, Z_A, Z_B) \) and in particular for the proof of (1)–(3), see Aventini and Seiler [2], Combes, Duclos, and Seiler [17], and Morgan and Simon [3]. The proof of (4) is briefly sketched in Proposition III.1 (2) as an easy application of the Hunziker–Vock [18] stability theorem. A proof of (5) has been given by Harrell [14] with some explicit estimates, and (6) has been proved by Morgan and Simon [3].

(2) The perturbation expansion is generated as follows (see, e.g., Morgan and Simon [3]): for \( |x| < R \), we have \( |x - R\hat{e}|^{-1} = \sum_{n=0}^{\infty} M_n(x) R^{-n-1} \), \( M_n(x) = |x|^n P_n(\cos \theta) \), \( \cos \theta = \langle x, \hat{e} \rangle / |x| \), where \( P_n(\cdot) \) is the \( n \)th Legendre polynomial. Then the unperturbed operator is \( H_0(Z_A) \), and the perturbation is by definition

\[-Z_B \sum_{n=0}^{\infty} M_n(x) \rho^{n+1}, \quad |x| < \rho^{-1}; \quad 0, \quad |x| \geq \rho^{-1} \].

The expansion obtained through
The ordinary Rayleigh–Schrödinger perturbation theory in \( \rho = 1/R \) near \( E(Z_A) \) is by definition the \( 1/R \) expansion.

(3) The Hamiltonian for \( H^+_2 \) is completely decomposed by the magnetic and parabolic quantum numbers, conventionally denoted respectively by integers \( m \), \( n_1 = j \geq 0 \) and \( n_2 = k \geq 0 \). The separability in elliptic coordinates detailed below implies that in any subspace of given \( m, j, k \) the eigenvalues of \( H(\rho, Z_A) \) come in asymptotically degenerate doublets for \( \rho \) sufficiently small, and gap estimates and asymptotic expansions analogous to those of (5) and (6) hold. The precise statements will be formulated below.

The well-known separability of \( H(\rho, Z_A, Z_B) \) in elliptic (more precisely, prolate spheroidal) coordinates goes back to Jacobi [19], who discovered its classical analogue to prove the complete integrability of the corresponding Hamilton–Jacobi equation. A thorough discussion of this problem and of its application to the Bohr–Sommerfeld quantization can be found in Born [20] (see also Strand and Reinhardt [21] for a modern analysis of the Bohr–Sommerfeld theory of \( H^+_2 \)). Let us now review the formulation of the Schrödinger eigenvalue problem

\[ H(\rho, Z_A, Z_B) \psi = E \psi \]

in elliptic coordinates. Standard references for this are Landau and Lifshitz [1] and Komarov et al. [22]. Set

\[
\begin{align*}
\xi &= \rho(|x| + |x - R\hat{e}|), \quad 1 \leq \xi \leq +\infty, \\
\eta &= \rho(|x| - |x - R\hat{e}|), \quad -1 \leq \eta \leq 1, \\
\phi &= \arctan(x_3/x_2), \quad 0 \leq \phi < 2\pi,
\end{align*}
\]

inverted as

\[
\begin{align*}
x_1 &= R\eta, \\
x_2 &= R \sqrt{(1 - \eta^2)(\xi^2 - 1)} \cos \phi, \\
x_3 &= R \sqrt{(1 - \eta^2)(\xi^2 - 1)} \sin \phi.
\end{align*}
\]

Since the Laplace operator in the variables \((\xi, \eta, \phi)\) has the form

\[
\Delta = 4\rho^2(\xi^2 - \eta^2)^{-1} \left[ \frac{\partial}{\partial \xi} (\xi^2 - 1) \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} (1 - \eta^2) \frac{\partial}{\partial \eta} \right.
\]

\[
\left. + \frac{\xi^2 - \eta^2}{(\xi^2 - 1)(1 - \eta^2)} \frac{\partial^2}{\partial \phi^2} \right]
\]

(see, e.g., Magnus Oberhettinger and Soni [23]), setting

\[ \psi(x) = e^{im\phi} \Phi_1(\xi) \Phi_2(\eta), \quad \pm m = 0, 1, 2, \ldots, \]

\[ (2.3) \]
we formally see that $\Psi$ satisfies $H(\rho, Z_A, Z_B) \Psi = E\Psi$ iff
\[
\left[ -\frac{1}{2} \frac{d^2}{d\xi} (\xi^2 - 1) \frac{d}{d\xi} - \frac{1}{4} R^2 E(\xi^2 - 1) - \frac{1}{2} R(Z_A + Z_B) \xi 
+ \frac{1}{2} m^2 (\xi^2 - 1)^{-1} \right] \Phi_1(\xi) = -\alpha \Phi_1(\xi),
\]
\[
\left[ -\frac{1}{2} \frac{d^2}{d\eta} (1 - \eta^2) \frac{d}{d\eta} - \frac{1}{4} R^2 E(1 - \eta^2) + \frac{1}{2} R(Z_A - Z_B) \eta 
+ \frac{1}{2} m^2 (1 - \eta^2)^{-1} \right] \Phi_2(\eta) = \alpha \Phi_2(\eta)
\]
(2.4)
for some separation constant $\alpha(m, R) \in \mathbb{R}$. The rest of this section is devoted to implementing this formal procedure so as to make transparent at the same time how the $1/R$ expansion is generated within the context of the separated equations. Set
\[
E = \frac{1}{2} \gamma^{-2}, \quad r = R \gamma^{-1}, \quad \tau = r^{-1},
\]
\[
\beta_1 = \frac{1}{2} \gamma (Z_A + Z_B) - \alpha \tau, \quad \beta_2 = \frac{1}{2} \gamma (Z_A - Z_B) + \alpha \tau
\]
(2.5)
and note the relations
\[
\beta_1 + \beta_2 = \gamma Z_A; \quad \frac{1}{2} \gamma (Z_A + Z_B) + \alpha \tau = \gamma (Z_A + Z_B) - \beta_1;
\]
\[
\frac{1}{2} \gamma (Z_A - Z_B) - \alpha \tau = \gamma (Z_A - Z_B) - \beta_2.
\]
(2.6)
Then, upon first rescaling the unknown functions
\[
\Phi_1(\xi) \mapsto (\xi^2 - 1)^{-1/2} \Phi_1(\xi), \quad \Phi_2(\eta) \mapsto (1 - \eta^2)^{-1/2} \Phi_2(\eta)
\]
(2.7)
and then translating and rescaling the variables $\xi$ and $\eta$,
\[
u = r(\eta - 1), \quad v = r(? + 1),
\]
(2.8)
Eqs. (2.1) become
\[
t_m(\beta_1, \beta_2, Z_A, Z_B, \tau) f(u) = 0, \quad s_m(\beta_1, \beta_2, Z_A, Z_B, \tau) g(v) = 0,
\]
(2.9)
where
\[
f(u) = [(\tau u + 1)^2 - 1]^{1/2} \Phi_1(\tau u + 1),
\]
\[
g(v) = [1 - (\tau v - 1)^2]^{1/2} \Phi_2(\tau v - 1),
\]
(2.10)
\[ t_m(\cdot) = \frac{d^2}{du^2} + \frac{1}{4} \frac{\beta_1}{u} + m^2 - 1 \frac{1}{4u^2} + \left[ \frac{(\beta_1 + \beta_2) Z_A^{-1}(Z_A + Z_B) - \beta_1}{u + 2r} + \frac{m^2 - 1}{4} \frac{1}{(u + 2r)^2} - \frac{1}{u(u + 2r)} \right], \]
\[ 0 \leq u < +\infty, \]
\[ s_m(\cdot) = \frac{d^2}{dv^2} + \frac{1}{4} \frac{\beta_2}{v} - m^2 - 1 \frac{1}{4v^2} + \left[ \frac{-\beta_2 - Z_A^{-1}(Z_A - Z_B)(\beta_1 + \beta_2)}{2r - v} + \frac{m^2 - 1}{4} \frac{2}{(v(2r - v))} + \frac{1}{(2r - v)^2} \right], \]
\[ 0 \leq v \leq 2r \]  
(2.11)

(2.12)

(u and v were called \( x_1 \) and \( x_2 \) in [14]). We then have

**Proposition II.2.** For \( \pm m = 0, 1, 2, \ldots \), let \( T_m(\beta_1, \beta_2, Z_A, Z_B, \tau) \), \( S_m(\beta_1, \beta_2, Z_A, Z_B, \tau) \), \( (\beta_1, \beta_2, Z_B) \in \mathbb{R}, (Z_A, \tau) \in \mathbb{R}^+ \) be the operator families in \( L^2(0, \infty) \), \( L^2(0, 2r) \), respectively, defined as the action of \( t_m(\cdot) \) on \( D(T_m(\cdot)) = \{ H^2(0, \infty) \cap H^1_0[0, \infty) \}, \mid m \mid > 0; H^2(0, + \infty) \} \) with the boundary condition \( f(u) = O(u^{1/2}) \) as \( u \downarrow 0 \) for \( m = 0 \), \( D(S_m(\cdot)) = \{ H^2(0, 2r) \cap H^1_0[0, 2r], \mid m \mid > 0; H^2(0, 2r) \} \) with boundary conditions \( f(v) = O(v^{1/2}), v \downarrow 0 \), \( f(v) = O((2r - v)^{1/2}), v \uparrow 2r, \) for \( m = 0 \), respectively. Then:

1. \( T_m(\cdot), S_m(\cdot) \) are self-adjoint and bounded below.
2. \( \sigma_{<\ast}(T_m(\cdot)) = \sigma_{<\ast}(S_m(\cdot)) = \left[ \frac{1}{4}, + \infty \right); \sigma_{\ast}(S_m(\cdot)) = \phi. \)
3. For any fixed \( (m, j, k) \) the eigenvalues \( \lambda(m, j, k; \beta_1, \beta_2, Z_A, Z_B, \tau) \) of \( T_m(\cdot) \) and \( \mu(m, k; \beta_1, \beta_2, Z_A, Z_B, \tau) \) of \( S_m(\cdot) \) are jointly continuously locally differentiable functions of the variables \( \beta_1, \beta_2, Z_A, Z_B, \tau \).
4. Assume that the equation \( \mu(m, k; \beta_1, \beta_2, Z_A, Z_B, \tau) = 0 \) can be solved near any given \( \overline{\tau} > 0 \) to yield a family of locally \( C^1 \) implicit functions \( \tau \mapsto \beta_2(m, k; \beta_1, Z_A, Z_B, \tau) \), \( (m, k; \beta_1, Z_A, Z_B) \) fixed, and that the equation \( \lambda(m, j; \beta_1, \beta_2(m, k; \beta_1, Z_A, Z_B, \tau); Z_A, Z_B, \tau) = 0 \) can be similarly solved to yield a family of locally \( C^1 \) implicit functions \( \tau \mapsto \beta_1(m, j; Z_A, Z_B, \tau) \), \( (m, j, k), (Z_A, Z_B) \) fixed. Set

\[ \gamma(m, j, k; Z_A, Z_B, \tau) = Z_A^{-1} [\beta_1(\cdot, \tau) + \beta_2(\cdot, \beta_1(\cdot, \tau), \cdot, \tau)] \]  
(2.13)

and assume that \( \tau \mapsto \gamma(\cdot, \tau)^{-1} \tau \) is locally invertible near any given \( \tau > 0 \), \( (m, j, k), (Z_A, Z_B) \) fixed. Let \( \rho \mapsto \Gamma(m, j, k; Z_A, Z_B; \rho) \) be the inverse function of \( \tau \mapsto \gamma(\cdot, \tau)^{-1} \tau \). Then the function

\[ E(m, j, k; Z_A, Z_B, \rho) = \frac{Z_A^2}{2} \left[ \gamma(m, j, k; Z_A, Z_B, \Gamma(m, j, k; Z_A, Z_B; \rho)) \right]^{-2} \]  
(2.14)

is an eigenvalue of \( H(\rho, Z_A, Z_B). \)
(5) Conversely, let $\rho \mapsto E(\rho, Z_A, Z_B)$ be an eigenvalue of $H(\rho, Z_A, Z_B)$. Then for one and only one triple $(m, j, k)$, $\pm m, j, k = 0, 1, \ldots$, the equations $\lambda(m, j, k; \beta_1, \beta_2, Z_A, Z_B, \tau) = 0$, $\mu(m, j, k; \beta_1, \beta_2, Z_A, Z_B, \tau) = 0$ can be solved near any given $\tau > 0$ to yield the pair of locally $C^1$ implicit functions $\tau \mapsto \beta_2(m, k; \beta_1, Z_A, Z_B, \tau)$, $\tau \mapsto \beta_1(m, j, k; Z_A, Z_B, \tau)$ such that $\tau \gamma(m, j, k; Z_A, Z_B, \tau)^{-1}$, $\gamma$ defined by (2.13), is invertible and $E(\rho, Z_A, Z_B)$ admits the representation (2.14).

Remarks. (1) Assertion (4) holds unchanged if the implicit functions are unraveled in the opposite order.

(2) The numbers $(m, j, k)$ have the meaning of magnetic and parabolic quantum numbers, respectively. In fact, letting $R \to \infty$ in (2.1) we have

$$R \xi - R = |x| - x_i + O(\rho), \quad R \eta + R = |x| + x_i + O(\rho),$$

which means that $\xi$ and $\eta$ become the usual parabolic coordinates (see, e.g., Landau and Lifshitz [1, Sect. 37]) up to rescaling and translation. Therefore, the natural number $n = |m| + j + k + 1$ has the meaning of principal quantum number.

(3) For $\tau = 0$ we recover the unperturbed operator $H_0(Z_A)$ in the following way: denote by $t_m^0(\beta)$ the differential expression obtained by setting formally $\tau = 0$ in (2.11) or, equivalently, (2.12):

$$t_m^0(\beta) \equiv t_m(\beta, 0) \equiv s_m(\beta, 0) = \frac{d^2}{du^2} + \frac{1}{4} - \beta u^{-1} + \frac{m^2 - 1}{4u^2},$$

$$0 \leq u < \infty. \quad (2.15)$$

Then the operator family $T_m^0(\beta) = T_m(\beta, 0)$ in $L^2(0, \infty)$ defined as the action of (2.15) on $D(T_m(\cdot))$ enjoys properties (1)–(3) above. Denote by $\lambda(m, j, k), |m|, j = 0, 1, \ldots$ the eigenvalues of $T_m^0(\beta)$. Then it is well known that $\lambda(m, j, k) = 0$ iff $\beta = \beta(m, j) = j + (|m| + 1)/2$, because the confluent hypergeometric equation $-\psi'' - \beta u^{-1}\psi + \frac{1}{4}\psi + ((m^2 - 1)/4u^2)\psi = 0$ admits solutions regular at 0 and $L^2$ at $+\infty$ iff $\beta = \beta(m, j)$ (see, e.g., Buchholz [24]). The corresponding (normalized) eigenfunctions are

$$\left[\frac{i}{(i + |m|)!3(|m| + 1 + 2i)}\right]^{1/2} u^{|m| + 1/2} e^{-u^2} L_m^{|m| + i}(u),$$

where $L_m^{|m|}(\cdot)$ are the Laguerre polynomials. Then we see at once that $\beta(m, j) + \beta(m, k) = \gamma(m, j, k) = i + k + |m| + 1$, and

$$\sigma_{\beta}(H_0(Z_A)) = \bigcup_{|m|, i, k = 0}^{\infty} -\frac{1}{2} Z_A^2 \gamma(m, i, k)^{-2}, \quad (2.16)$$

which is equivalent to assertions (4) and (5) because in this case $\gamma(\cdot, \tau)$ is $\tau$-independent.
Proof. Assertions (1) and (2) are well known (see, e.g., Kato [16] for \(m \neq 0\) or Dunford and Schwartz [25] for \(m = 0\)). Statement (3) follows by standard arguments of regular perturbation theory (worked out in detail for the case of the non-separated operator in Combes, Duclos and Seiler [17]). We prove (4) and (5). Denote by \(f(u, m, j; \beta_1, \beta_2; Z_A, Z_B, \tau)\) and \(g(v, m, k; \beta_1, \beta_2; Z_A, Z_B, \tau)\) the eigenvectors corresponding respectively to \(\lambda(m, j; \tau)\) and \(\mu(m, k; \tau)\). Then the function

\[
(x; m, j, k; Z_A, Z_B; \rho) \rightarrow \Psi(x; m, j, k; Z_A, Z_B; \rho) = e^{im\arctan(x_3/x_2)}[\Gamma(m, j, k; Z_A, Z_B; \rho)[|x| + |x - R\tilde{e}| - 1]^{-1/2} \cdot [\Gamma(\cdot)[\rho(|x| - |x - R\tilde{e}|) + 1]]^{-1/2} f(\Gamma(\cdot)[\rho(|x| - |x - R\tilde{e}|) + 1]; m, j, \beta_1(\cdot, \Gamma(\cdot)), \beta_2(\cdot, \Gamma(\cdot)), \cdot, \Gamma(\cdot)) \cdot g(\Gamma(\cdot)[\rho(|x| - |x - R\tilde{e}|) + 1]; m, k, \beta_1(\cdot, \Gamma(\cdot)), \beta_2(\cdot, \Gamma(\cdot)), \cdot, \Gamma(\cdot))
\]

belongs to \(H^2(\mathbb{R}^3)\) and satisfies

\[
H(\rho, Z_A, Z_B) \Psi = E\Psi
\]

with \(E\) given by (2.14) by direct inspection by virtue of (2.1)–(2.12). Conversely, to see (5), let \((x, \rho; Z_A, Z_B) \rightarrow \Psi(x, \rho; Z_A, Z_B)\) be an eigenvector of \(H(\rho, Z_A, Z_B)\); \(H(\rho, Z_A, Z_B) \Psi = E\Psi\). The change of variables (2.1)–(2.2) induces the direct sum decomposition

\[
L^2(\mathbb{R}^3) = \bigoplus_{m = -\infty}^{+\infty} L_m, \quad L_m = L^2(\Omega; d\omega) \otimes e^{im\phi},
\]

where

\[
\Omega = \{(\xi, \eta): 1 < \xi < \infty, -1 < \eta < 1\};
\]

\[
d\omega = (\xi^2 - \eta^2) \, d\xi \, d\eta.
\]

Now \(L_m\) reduces \(H(\rho, Z_A, Z_B)\) for all \(m\). Hence we can write

\[
\Psi = \sum_{m = -\infty}^{+\infty} e^{im\phi} \Phi(m; \xi, \eta; E(m))
\]

with

\[
\left\| (\eta^2 - \xi^2)^{-1} \left[ \frac{\partial}{\partial \xi} (\xi^2 - 1) \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} (1 - \eta^2) \frac{\partial}{\partial \eta} \right. \left. + \frac{m^2 (\xi^2 - \eta^2)}{(\xi^2 - 1)(1 - \eta^2)} \right] \Phi(m; \xi, \eta; E(m)) \right\|_{L^2(\Omega, d\omega)} < \infty
\]

(2.21)
and

\[ -4\rho^2(\xi^2 - \eta^2)^{-1} \left[ \frac{\partial}{\partial \xi} (\xi^2 - 1) \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} (1 - \eta^2) \frac{\partial}{\partial \eta} \right] \Phi(m; \xi, \eta; E(m)) - 2\rho Z_A(\xi + \eta)^{-1} \Phi(m; \xi, \eta; E(m)) \\
+ \frac{m^2(\xi^2 - \eta^2)}{(\xi^2 - 1)(1 - \eta^2)} \right] \Phi(m; \xi, \eta; E(m)) - 2\rho Z_B(\xi - \eta)^{-1} \Phi(m; \xi, \eta; E(m)) = E(m) \Phi(m; \xi, \eta; E(m)) \quad (2.22) \]

for some \( m \in \mathbb{Z} \), i.e., we have

\[ H(\rho, Z_A, Z_B) = \bigoplus_{m = -\infty}^{+\infty} H_m(\rho, Z_A, Z_B), \quad (2.23) \]

where \( H_m(\rho, Z_A, Z_B) \) is the self-adjoint operator on \( L^2(\Omega, d \omega) \) defined as the action of the left side of (2.22) on all functions in \( L^2(\Omega; d \omega) \) satisfying (2.21). Therefore, there is an \( m \in \mathbb{Z} \) such that \( E = E(m) \in \sigma_d(H_m) \). On the other hand the map \((Qf)(\xi, \eta) = (\xi^2 - \eta^2)^{1/2} f(\xi, \eta)\) is unitary from \( L^2(\Omega; d \omega) \) to \( L^2(\Omega; d \xi \, d \eta) \) and therefore \( E(m) \) is an eigenvalue of \( H_m \) if and only if 0 is eigenvalue of \( QH_m Q^{-1} \), defined as the action of

\[ \begin{align*}
-\frac{1}{2} \frac{\partial}{\partial \xi} & (\xi^2 - 1) \frac{\partial}{\partial \xi} - \frac{1}{2} \frac{\partial}{\partial \eta} (1 - \eta^2) \frac{\partial}{\partial \eta} - \frac{1}{4} R^2 E[(\xi^2 - 1) + (1 - \eta^2)] \\
-\frac{1}{2} R(Z_A + Z_B) \xi + \frac{1}{2} (Z_A + Z_B) \eta + \frac{1}{2} m^2 [(\xi^2 - 1)^{-1} + (1 - \eta^2)^{-1}] 
\end{align*} \]

on \( QD(H_m(\cdot)) \). In turn, we have

\[ QH_m(\cdot) Q^{-1} = UT_m(\cdot) U^{-1} \otimes I_{L^2(0,2)} + I_{L^2(0,\infty)} \otimes VS_m(\cdot) V^{-1}, \quad (2.24) \]

where \( T_m(\cdot) \) and \( S_m(\cdot) \) are defined above, and \((Uf)(\xi) = (\xi^2 - 1)^{-1/2} f(r(\xi - 1)), (Vg)(\eta) = (1 - \eta^2)^{-1/2} g(r(\eta + 1))\). Therefore (2.24) and the theorem on the spectrum of tensor products (see, e.g., Reed and Simon [15, Theorem VIII.33]) precisely characterize the union of the sets of values of \( E(m) \) such that \( QH_m(\cdot) Q^{-1} \) has the eigenvalue 0, in the form (2.14).

We can now formulate the \( 1/R \) expansion via the separated equations.

**Proposition II.3.** Consider the eigenvalues \( \lambda(m, j; \beta_1, \beta_2; Z_A, Z_B; \tau) \equiv \lambda(\cdot, \tau) \) of \( T_m(\cdot) \), and the eigenvalues \( \mu(m, k; \beta_1, \beta_2; Z_A, Z_B; \tau) \equiv \mu(\cdot, \tau) \) of \( S_m(\cdot) \). Denote once again by \( \lambda(m, j; \beta) \equiv \lambda(\cdot) \) the eigenvalues of \( T_m(\beta) \). Then:
(1) For any fixed $m, j, \beta_1 > 0, \beta_2 > 0, Z_A > 0,$ and $Z_B \in \mathbb{R},$ the functions $\lambda(\cdot, \tau)$ and $\mu(\cdot, \tau)$ admit asymptotic expansions near $\lambda(\cdot)$ to all orders in $\tau/2 > 0$ as $\tau \downarrow 0$:

\[
\lambda(\cdot, \tau) \sim \lambda(\cdot) + \sum_{m=1}^{\infty} A_n(\cdot)(\tau/2)^n, \\
\mu(\cdot, \tau) \sim \lambda(\cdot) + \sum_{n=1}^{\infty} B_n(\cdot)(\tau/2)^n.
\]  
(2.25)  

(2.26)  

The coefficients $A_n(m, j, \beta_1, \beta_2, Z_A, Z_B)$, $B_n(m, k; \beta_1, \beta_2, Z_A, Z_B)$ are given by Rayleigh–Schrödinger perturbation theory in $L^2(0, +\infty)$ in the following way: the unperturbed operator is $T_m^0(\beta_1)$, $T_m^0(\beta_2)$, respectively, and the perturbation is the maximal multiplication operator by $F(u, \cdot, \tau)$ in case (2.25), $G(v, \cdot, \tau)$ in case (2.26), respectively. Here

\[
F(u, \cdot, \tau) = \sum_{n=1}^{\infty} F_n(u, \cdot)(\tau/2)^n,
\]  
(2.27)  

\[
F_n(u, \cdot) = 0, \quad u > 2r, \\
= \left(\beta_1 + \beta_2 \right) Z_A^{-1}(Z_A + Z_B) - \beta_1 \right) (-1)^n u^{n-1} + \frac{m^2 - 1}{4} (-1)^n (n + 1) u^{n-2}, \quad u < 2r,
\]  
(2.28)  

\[
G(v, \cdot, \tau) = \sum_{n=1}^{\infty} G_n(v, \cdot)(\tau/2)^n,
\]  
(2.29)  

\[
G_n(v, \cdot) = 0, \quad v \geq 2r, \\
= - \left(\beta_2 - Z_A^{-1}(Z_A - Z_B) (\beta_1 + \beta_2) \right) v^{n-1} + \frac{m^2 - 1}{4} (n + 1) v^{n-2}, \quad v < 2r.
\]  
(2.30)  

(2) The functions $\lambda(m, j, \beta_1, \beta_2, \cdot, \tau)$, $\mu(m, k; \beta_1, \beta_2, \cdot, \tau)$ are $C^\infty$ in $(\beta_1, \beta_2, \tau)$ in a neighborhood of $\beta(m, j) \times \beta(m, k) \times \bar{\tau}$, $(|m|, j, k) = 0, 1, ..., \bar{\tau} > 0$. The functions $\tau \mapsto \beta_2(m, k, \cdot, \tau)$ and $\tau \mapsto \beta_1(m, j, k, \cdot, \tau)$ are $C^\infty$ near any given $\bar{\tau} > 0$, and admit an asymptotic expansion to all orders as $\tau \downarrow 0$:

\[
\beta_2(m, k, \cdot, \tau) \sim \beta(m, k) + \sum_{n=1}^{\infty} L_n(m, k, \cdot)(\tau/2)^n, \\
\beta_1(m, j, \cdot, \tau) \sim \beta(j, k) + \sum_{n=1}^{\infty} M_n(m, j, \cdot)(\tau/2)^n.
\]  
(2.31)  

(2.32)  

The functions $\rho \mapsto \Gamma(m, j, k; \rho)$ and $\rho \mapsto E(m, j, k, \rho)$ (given by (2.14)) are $C^\infty$ near any given $\bar{\rho} > 0$ and admit an asymptotic expansion to all orders as $\rho \to 0$. The asymptotic expansion for $E(m, j, k; \rho)$ coincides with the 1/R expansion near the
eigenvalue of $H_0(Z_A)$ of magnetic quantum number $m$ and parabolic quantum numbers $(j, k)$ written as

$$E(m, j, k; \rho) \sim E(m, j, k) + \sum_{n=1}^{\infty} E_n(m, j, k) \rho^n.$$ (2.33)

Remarks. (1) Remark (3) after Proposition II.1 can now be more precisely formulated as follows: for any eigenvalue $E(m, j, k) = -\frac{1}{2} Z_A^2 (|m| + j + k + 1)^{-2}$ of $H_0(Z_A)$, $|m|, j, k = 0, 1, \ldots$ fixed, and any open interval $I$ containing only $E(m, j, k)$, there is $M(m, j, k)$ such that for $\rho < M$ there are precisely two eigenvalues $E_\pm(m, j, k; \rho)$ of $H(\rho, Z_A)$ in $I$. Furthermore, we have [13]

$$\Delta E(m, j, k; \rho) = E_+(m, j, k; \rho) - E_-(m, j, k; \rho) = O(m, j, k; \rho^{-(2k + |m| + 1)} \exp(-1/\rho (j + k + |m| + 1))),$$ (2.34)

where, here and elsewhere, $O(m, j, k; x)$ stands for order $x$ with constant depending on $(m, j, k)$.

(2) Completely analogous statements hold for $S_m(\beta_1, \beta_2, Z_A = Z_B; \tau) = S_m(\beta_2, Z_A, \tau)$: given any eigenvalue $\mu(m, k; \beta_2, Z_A)$ of $S_m(\beta_2, 0)$ (defined by (2.15)) and any open interval $I$ as above, there is a constant $M(m, k)$ such that for $\tau < M$, $S_m(\beta_2, Z_A, \tau)$ has exactly two eigenvalues $\mu_\pm(m, k, \beta_2, Z_A, \tau)$ in $I$, such that

$$\Delta \mu(m, k; \beta_2, Z_A; \tau) = \mu_+(\cdot) - \mu_-(\cdot) = O(m, k; \tau^{-(2k + |m|)} e^{-1/\tau})$$ (2.35)

uniformly on compacts in $(\beta_2, Z_A) \in \mathbb{R}^+$. Hence, upon putting the implicit relation in explicit form for each fixed $\pm m, k = 0, 1, \ldots$ there are $\beta_2^{\pm}(m, k; Z_A, \tau) \to \beta(m, k; Z_A)$ as $\tau \to 0$ such that

$$\Delta \beta_2(m, k; Z_A) = \beta_2^+(\cdot) - \beta_2^-(\cdot) = O(m, k; \tau^{-(2k + |m| + 1)} e^{-1/\tau})$$ (2.36)

uniformly on compacts in $Z_A \in \mathbb{R}^+$. For the proof of (2.35), (2.36), see Harrell [13].

Proof. Assertion (1) can be proved by well-known arguments of singular perturbation theory (we omit the details because they have been worked out in the present case by Morgan and Simon in the more general context of the non-separated formalism). A statement stronger than (2), namely, local analyticity in $(\beta_1, \beta_2, \tau)$ can be proved by exactly the same argument as in Proposition III.3(1) for the function $\lambda(\cdot, \beta_1, \beta_2, \tau)$. If we now observe that by the unitary rescaling, $(V(r)f)(v) = r^{1/2}f(\tau v)$ mapping $L^2(0, 2r)$ onto $L^2(0, 2)$ one-to-one, $\mu(\cdot, \beta_1, \beta_2, \tau)$ is an eigenvalue of $V(r) S_m(\cdot) V(r)^{-1}$, which is the action

$$r^{-2} \left[ -\frac{d^2}{dv^2} + \frac{1}{4} r^2 - \frac{r \beta_2}{v} + \frac{m^2 - 1}{4 v^2} + \frac{\beta_2 - Z_A^{-1}(Z_A - Z_B)(\beta_1 + \beta_2)}{2 - v} \right] + \frac{m^2 - 1}{4} \left( \frac{2}{v(2 - v)} + \frac{1}{(2 - v)^2} \right)$$
on $V(r) D(S_m(\cdot))$, we get by the same argument also the local analyticity of $(\beta_1, \beta_2, \tau) \mapsto \mu(\cdot, \beta_1, \beta_2, \tau)$ because it is immediately seen that $V(r) D(S_m(\cdot))$ is independent of $(\beta_1, \beta_2, \tau)$. The implicitly defined functions $\tau \mapsto \beta_1(m, j, k; \tau), \tau \mapsto \beta_2(m, k; \tau)$ exist by Proposition II.2(4) and are thus locally $C^\infty$. Hence the validity of the asymptotic expansions (2.31), (2.32) is a consequence of (1) and of the implicit-function theorem. The functions $\tau \mapsto \gamma(m, j, k; \tau)^{-1 \tau}$ are invertible again by II.2(4), and $I(m, j, k; \rho)$ and $E(m, j, k; \rho)$ are locally $C^\infty$ and admit asymptotic expansions to all orders once again by the implicit-function and local-invertibility theorems, given (2.13), (2.14), (2.31), and (2.32). Finally, we note that the expansion for $E(\cdot, \rho)$ generated via (2.31), (2.32), (2.13), and (2.14) coincides with the $1/R$ expansion because a function can have at most one asymptotic expansion.

III. Stability, Analyticity, and Summability

The main purpose of this section is to identify the Borel sum of the $1/R$ expansion for $H^*_2$ near any eigenvalue $E(m, j, k; Z_A)$ of $H_0(Z_A)$ of magnetic quantum number $m$ and parabolic quantum numbers $(j, k)$.

To this end, we consider two distinct cases in the two-center operator family $H(\rho, Z_A, Z_B)$, which we now describe in order also to establish some further notation utilized throughout the rest of this paper.

Case A (the $H^*_2$ problem): $\rho > 0, Z_A = Z_B = 1$.

Case B: $\rho = -\rho', \rho' > 0, Z_A = 1, Z_B = -1$.

We denote $H(\rho, 1, 1) \equiv H(\rho), H(\rho', 1, -1) \equiv H'(\rho')$. The physical interpretation of $H'(\rho')$ was mentioned in Section I, and its relevant mathematical properties are summarized as follows:

**Proposition III.1.** Let $H'(\rho')$ be the operator in $L^2(\mathbb{R}^2)$ defined as the action of $-\frac{1}{2} A - |x|^{-1} + |x + \rho'\hat{e}|^{-1}$ on $H^2(\mathbb{R}^2)$. Then $H'(\rho')$ enjoys properties (1), (2) of Proposition II.1, and, furthermore:

1. Each eigenvalue $E$ of $H_0(Z_A = 1)$ is stable (in the sense of Kato [16, Sect. VIII.1.5]) as an eigenvalue $E'(\rho')$ of $H'(\rho')$ as $\rho' \downarrow 0$.

2. Let $E'(\rho')$ be the ground state of $H'(\rho')$, and $E'(\rho') \sim E + \sum_{n=1}^\infty E_n(\rho')^{n}$ be its $\rho'$ expansion near $E$, the ground state of $H_0(Z_A = 1)$. Then $E_n = (-1)^n E_n$, where $E_n$ are the coefficients of the $1/R$ expansion for $H^*_2$ near $E$.

**Remark.** We will see below that actually $E_n(m, j, k) = (-1)^n E_n(m, j, k)$ for each triple of quantum numbers $(|m|, j, k) = 0, 1, 2, \ldots$.

**Proof.** Assertion (1) is an immediate application of the Hunziker–Vock stability theorem [18]: in fact,

$$\|x + \rho'\hat{e}|^{-1}\|_{L^2(\mathbb{R}^2)} \to 0$$
as \( \rho' \to 0 \), and this implies (see again Ref. [8, Lemma 1.2]) that \( H'(\rho') \) converges in strong-resolvent sense to \( H_0(Z_A) \) as \( \rho' \to 0 \). Furthermore, given \( x \mapsto \chi(x) \in C_0^\infty(\mathbb{R}^3) \), \( \chi(x) = 1, \ |x| \leq 1 \); \( \chi(x) = 0, \ |x| \geq 2 \), and setting \( M_n(x) = 1 - \chi(x/n) \), we have \( \lim_{n \to \infty} \text{dist}(E, W_n(\rho')) > 0 \) uniformly with respect to \( \rho' \) for all \( E < 0 \). Here

\[
W_n(\rho') = \{ z : z = \langle M_nu, H'(\rho') M_nu \rangle ; \ u \in C_0^\infty(\mathbb{R}^3); \ |u| = 1 \}.
\]

In fact, \( \langle -\frac{1}{2} A M_nu, M_nu \rangle + \langle |x + \rho'\ell|^{-1} M_nu, M_nu \rangle \geq 0 \) independently of \( n \), and \( \langle -|x|^{-1} M_nu, M_nu \rangle \geq -1/n \). Since all eigenvalues of \( H_0 \) are negative, the conditions of [18, Theorem 1.1] are satisfied and (1) is proved. Assertion (2) is trivial given Remark (2) after Proposition II.1.

Let us now specialize the general formalism of Propositions II.2, II.3 to the Cases A and B. We use the convention of denoting each quantity relative to \( H'(\rho') \) with a prime on the corresponding quantity relative to \( H(\rho) \). More specifically, considering the operators \( T_m(\cdot) \) and \( S_m(\cdot) \) defined in Proposition II.2, we set for Case A (the \( H_2^+ \) system \( Z_A = Z_B = 1 \))

\[
T_m(\beta_1, \beta_2; 1, 1, \tau) = T_m(\beta_1, \beta_2, \tau), \tag{3.1}
\]

\[
S_m(\beta_1, \beta_2, 1, 1, \tau) = S_m(\beta_2, \tau),
\]

because the differential expressions \( t_m(\cdot) \) and \( s_m(\cdot) \) simplify to

\[
t_m(\beta_1, \beta_2, \tau) = -\frac{d^2}{du^2} + \frac{1}{4} \frac{\beta_1}{u} + \frac{m^2 - 1}{4u^2} \frac{2\beta_2 + \beta_1}{u} + \frac{m^2 - 1}{4} ((u + 2r)^{-2} - 2u^{-1}(u + 2r)^{-1}) \tag{3.2}
\]

and

\[
s_m(\beta_2, \tau) = -\frac{d^2}{dv^2} + \frac{1}{4} \frac{\beta_2}{v} + \frac{m^2 - 1}{4v^2} \frac{\beta_2}{2r - v} + \frac{m^2 - 1}{4} (2v^{-1}(2r - v)^{-1} + (2r - v)^{-2}) \tag{3.3}
\]

For Case B, i.e., the operator \( H'(\rho') \) with \( Z_A = -Z_B = 1, \ \rho' = -\rho \), the separated operators are, respectively,

\[
T_m(\beta_1', \beta_2'; 1, -1, \tau') \equiv T_m(\beta_1', \tau'), \tag{3.4}
\]

i.e., the action on \( D(T_m) \) of the differential expression

\[
t_m'(\beta_1', \tau') = -\frac{d^2}{du^2} + \frac{1}{4} \frac{\beta'}{u} + \frac{m^2 - 1}{4u^2} + \frac{\beta_1'}{2r' + u} \tag{3.5}
\]

\[
+ \frac{m^2 - 1}{4} ((2r' + u)^{-2} - 2u^{-1}(2r' + u)^{-1}),
\]
and

\[ S_m(\beta_1', \beta_2'; 1, -1, \tau') \equiv S_m'(\beta_1', \beta_2'; \tau'), \]

i.e., the action on \( D(S_m) \) of the differential expression

\[
S_m'(\beta_1', \beta_2'; \tau') = -\frac{d^2}{dv^2} + \frac{1}{4} \frac{\beta_2'}{v} + \frac{m^2 - 1}{4v^2} + \frac{2\beta_1' + \beta_2'}{2r' - v} \\
+ \frac{m^2 - 1}{4} ((2r' - v)^{-2} + 2v^{-1}(2r' - v)^{-1}).
\]

The functions \( \lambda(m, j, \beta_1, \beta_2, \tau) \equiv \lambda(m, j, \beta_1, \beta_2, 1, 1, \tau), \mu(m, k, \beta_2, \tau) \equiv \mu(m, k; \beta_1, \beta_2, 1, 1, \tau), \beta_2(m, k; \beta_1'; \tau) \equiv \beta_2(m, k; \beta_1; 1, 1, \tau), \beta_1(m, j, k; \tau) \equiv \beta_1(m, j, k; 1, 1, \tau), \gamma(m, j, k; \tau) \equiv \gamma(m, j, k; 1, 1, \tau), \Gamma(m, j, k; \rho) \equiv \Gamma(m, j, k; 1, 1, \rho) \), and their primed counterparts have the same meaning as in Section II. We denote again by \( \lambda(m, j, \beta) \) the eigenvalues of \( T_m^0(\beta) \). The functions

\[
E(m, j, k; \rho) = -\frac{1}{2} \left[ \gamma(m, j, k; \Gamma(m, j, k; \rho)) \right]^{-2}, \quad (|m|, j, k) = 0, 1, \ldots,
\]

\[
E'(m, j, k; \rho') = -\frac{1}{2} \left[ \gamma'(m, j, k; \Gamma'(m, j, k; \rho')) \right]^{-2},
\]

yield respectively the discrete spectra of \( H(\rho) \) and \( H'(\rho') \). Furthermore, formulae (2.27)-(2.30) together with their primed counterparts simplify to

\[
F_n(u, \beta_1, \beta_2) = 0, \quad u \geq 2r,
\]

\[
= (2\beta_2 + \beta_1)(-1)^n u^{n-1} + \frac{m^2 - 1}{4} (-1)^n (n + 1) u^{n-2}, \quad u < 2r,
\]

\[
F'_n(u, \beta_1') = 0, \quad u \geq 2r,
\]

\[
= \beta_1'(-1)^{n-1} u^{n-1} + \frac{(m^2 - 1)}{4} (-1)^n (n + 1) u^{n-2}, \quad u < 2r,
\]

and

\[
G_n(v, \beta_2) = 0, \quad v \geq 2r,
\]

\[
= -\beta_2 v^{n-1} + \frac{m^2 - 1}{4} (n + 1) v^{n-2}, \quad v < 2r,
\]

\[
G'_n(v, \beta_1, \beta_2') = 0, \quad v \geq 2r,
\]

\[
= (2\beta_2 + \beta_1) v^{n-1} + \frac{m^2 - 1}{4} (n + 1) v^{n-2},
\]

so that the expansions (2.25) and (2.26) for \( \mu(m, k; \beta_2, \tau) \) and \( \lambda(m, j, \beta_1, \beta_2, \tau) \) hold, together with their primed counterparts for \( \mu'(m, k; \beta_1, \beta'_2, \tau') \) and
\( \lambda(m, j, \beta'_1, \tau') \). We denote their coefficients by \( B_n(m, k; \beta_2) \), \( A_n(m, j; \beta_1, \beta_2) \), \( B'_n(m, k; \beta'_1, \beta'_2) \), and \( A'_n(m, j; \beta'_1) \), respectively. Analogously, we denote by \( L'_n(m, k) \) and \( M'_n(m, j, k) \) the coefficients of the primed counterparts of the asymptotic expansions (2.31) and (2.32), specialized in this way. Obviously, the \( r \)-dependence implicit in Eq. (3.10)–(3.13) does not affect the computations of the perturbation coefficients; because of the exponential decay of the unperturbed eigenfunction, it introduces only exponentially small corrections.

To get the above-mentioned result on the identification of the Borel sum of the \( 1/R \) expansion as a complex eigenvalue obtained by interconnecting \( H(\rho) \) and \( H'(\rho') \), the "double-well" operator \( S_m(\beta_2, \tau) \) in the finite interval \((0, 2r)\) has to be replaced by the analytic continuation up to \( \tau' = e^{\pm ir}, \tau > 0 \), of the "single-well" operator \( T'_m(\beta'_1, \tau') \), \( \tau' > 0 \), in the infinite interval \((0, +\infty)\). This mechanism, which identifies the Borel sum for \( \tau' > 0 \), is basically the same as that which gives rise to existence and Borel summability of resonances out of the separability in squared parabolic coordinates in the Lo Surdo–Stark effect [7]. A major difference is that here the "single-well" equation is that of Case B. Of course, the non-self-adjoint, stable problem having the same \( 1/R \) expansion as \( H'_+ \) can be immediately defined (see the subsequent proposition) within the separated formalism out the operators \( T'_m(\beta'_1, e^{-ir}), T_m(\beta'_1, \beta'_1(\tau e^{-\pi}), \tau) \) realized below. The result, whose proof is to be obtained in the course of this section, reads as follows:

**Theorem III.2.** Let \(|m|, j, k\) = 0, 1,\ldots be fixed. Then for any \( \mu = \mu(m, j, k) > 0 \) there are \( 0 < M = M(m, j, k) < \infty \) and \( 0 < M'_1(m, k) < \infty \) such that:

1. The implicitly defined functions \( \tau' \mapsto \beta'_1(m, k; \tau') \) exist as holomorphic functions of \( \tau' \) for \( 0 < M_1, |\arg \tau'| < \pi \), admit analytic continuation to the Riemann-surface sector \( \mathcal{G}(m, k) = \{ \tau': 0 < |\tau'| < M_1; |\arg \tau'| < \frac{\pi}{2} - \mu \} \) across the negative real axis, and \( \lim_{\tau' \to 0} \beta'_1(m, k; \tau') = \beta(m, k) = k + \frac{1}{2}(|m| + 1) \) as \( \tau' \to 0, \tau' \in \mathcal{G} \).

2. The implicitly defined functions \( \tau \mapsto \beta'_1(m, j; \beta'_1(m, k; \tau e^{-\pi})), \tau \), which will be denoted for convenience as \( \beta'_1(m, j, k; \tau) \), exists as holomorphic functions of \( \tau \) for \( 0 < |\tau| < M \), \( 0 < |\arg \tau| < \pi \), admit analytic continuation to the Riemann-surface sector \( \mathcal{D}(m, j, k) = \{ \tau: 0 < |\tau| < M; -\pi/2 + \mu < |\arg \tau| < \frac{\pi}{2} - \mu \} \) across the real axis, and \( \lim_{\tau \to 0} \beta'_1(m, j, k; \tau) = \beta(m, j) = j + \frac{1}{2}(|m| + 1) \) as \( \tau \to 0, \tau \in \mathcal{D}(m, j, k) \).

3. The functions \( \tau \mapsto \gamma_1(m, j, k; \tau) = \beta_1(m, j, k; \tau) + \beta'_1(m, k; \tau e^{-\pi}) \) are holomorphic for \( 0 < |\tau| < M \), \( 0 < |\arg \tau| < \pi \), and admit analytic continuation to \( \mathcal{D}(m, j, k) \) as above. The functions \( \gamma_1(m, j, k; \tau)^{-1} \) are invertible in \( \mathcal{D}(m, j, k) \); the inverse functions \( \rho \mapsto \Gamma_1(m, j, k; \rho) \) of \( \gamma_1(m, j, k; \tau)^{-1} \) are holomorphic for \( 0 < |\rho| < M \), \( 0 < |\arg \rho < \pi \), and admit analytic continuation to \( \mathcal{D}(m, j, k) \) as above.

4. The functions

\[ \rho \mapsto E_1(m, j, k; \rho) = -\frac{1}{2} \left[ \gamma_1(m, j, k; \Gamma_1(m, j, k; \rho)) \right]^{-2} \]  

and holomorphic for \( 0 < |\arg \rho < \pi \), admit analytic continuation to \( \mathcal{D}(m, j, k) \) as above, and have the same \( \rho = 1/R \) expansion as \( E(m, j, k; \rho) \).
(5) The $1/R$ expansion near any eigenvalue $E(m, j, k)$ of $H_0$ is Borel summable not to $E_+(m, j, k; \rho)$ or to $E_-(m, j, k; \rho)$, but to $E_1(m, j, k; \rho)$ for $0 < |\rho| < M$, $-\pi/2 + \mu < \arg \rho < \frac{1}{2} \pi - \mu$.

Remarks. (1) The definition of $\rho'$ as $e^{-i\alpha} \rho$ makes $\text{Im} E_1(\cdot, \rho) \leq 0$. The opposite choice of phase would have made $\text{Im} E_1(\cdot, \tau) \geq 0$.

(2) In terms of the Borel summability in the standard sense (see, e.g., Reed and Simon [15, Sect. XII.4]) statement (5) means that the $1/R$ expansion is Borel summable to $E_1(m, j, k; \rho)$ for $0 < \arg \rho < \pi$, $|\rho| < M$. Thus, for $\rho$ real $E_1(m, j, k; \rho)$ is determined from the Borel sum ((4)) and analytic continuation to the real axis. On the other hand, under the present conditions, the analytic continuation can be explicitly written in terms of the Nevanlinna modified representation of the Borel integral (for details see, e.g., Sokal [26]), namely,

$$E_1(m, j, k; e^{i\alpha} \rho) = R \int_0^\infty e^{-t e^{i\alpha} F_\alpha(t)} dt,$$

$$-\pi/2 + \mu < \alpha + \arg \rho < \frac{1}{2} \pi - \mu,$$

where $F_\alpha(t)$ is the Borel transform of the $1/R$ expansion computed at $\rho = t e^{i\alpha}$. Therefore statement (5) can be considered equivalent to (3.15).

(3) Statement (5), and hence also Remark (2) above, applies to the separation-constant eigenvalues as well. That is, the perturbation series (2.32) coincides with the perturbation series for $\beta'_1(\cdot, \tau e^{-i\alpha})$ and is Borel summable to that function and not to $\beta'_1(\cdot, \tau)$; the perturbation series (2.31) is Borel summable to $\beta_1(\cdot; \tau)$ and not to $\beta_1(\cdot; \beta'_1(\cdot, \tau), \tau)$; and the series for $\gamma$ is summable not to $\gamma(\cdot, \tau)$ but to $\gamma_1(\cdot; \tau)$.

(4) Interchanging the roles of $\rho$ and $\rho'$, a statement equivalent to (5) is that the $\rho'$ expansion for each eigenvalue $E'(m, j, k; \rho')$ of $H'(\rho')$ is Borel summable to $E_2(m, j, k; \rho') \equiv -\frac{1}{2} \left[ \gamma_2(m, j, k; \Gamma_2(m, j, k; \rho')) \right]^{-2}$. Here $\tau' \mapsto \gamma_2(m, j, k; \tau') = \beta'_1(m, j; \tau') + \beta'_2(m, k; \beta'_1(m, j; \tau'), e^{-i\tau'}$, and $\rho' \mapsto \Gamma_2(m, j, k; \rho')$ is the inverse function of $\tau'/\gamma_2(\cdot; \tau')$. Of course the remarks above apply also to this case.

(5) We will see in Proposition IV.1 that $\text{Im} E_1(\cdot, \rho)$ is non-zero for $\rho$ real and small. Since the $1/R$ expansion has real coefficients, the Borel summability implies its divergence.

The first step in proving Theorem III.2 is represented by the analysis of the operator families $T_m(\beta'_1, \tau')$, $T_m(\beta_1, \beta_2, \tau)$ for suitable complex values of the parameters. For $\theta \in \mathbb{C}$, $|\text{Im} \theta| < \pi/2$, set

$$p(u, m, \beta_1, \beta_2, \tau, \theta) = \frac{2\beta_1 + \beta_1}{e^{\theta u} + 2r}$$

$$+ \frac{m^2 - 1}{4} \left( (e^{\theta u} + 2r)^{-2} - 2e^{-\theta u} - (e^{\theta u} + 2r)^{-1} \right) \quad (3.16)$$
and
\[
q(u, m; \beta_1', \tau', \theta) = \frac{\beta_1'}{2r' + e^{\theta}u} \\
+ \frac{m^2 - 1}{4} ((e^{\theta}u + 2r')^{-2} - 2e^{-\theta}u^{-1}(e^{\theta}u + 2r')^{-1}).
\]
(3.17)

Hence, if we define the differential expressions
\[
t_m(\beta_1, \beta_2, \tau, \theta) = -e^{-2\theta} \frac{d^2}{du^2} + e^{-2\theta} \frac{m^2 - 1}{4u^2} - e^{-\theta} \frac{\beta_1}{u} \\
+ p(u, m, \beta_1, \beta_2, \tau, \theta) + \frac{1}{4}
\]
(3.18)
and
\[
t_m'(\beta_1', \tau', \theta) = -e^{-2\theta} \frac{d^2}{du'^2} + e^{-2\theta} \frac{m^2 - 1}{4u'^2} - e^{-\theta} \frac{\beta_1'}{u'} \\
+ q(u, m; \beta_1', \tau', \theta) + \frac{1}{4}
\]
(3.19)

by (3.4) and (3.6), we have
\[
t_m(\beta_1, \beta_2, \tau, 0) = t_m(\beta_1, \beta_2, \tau);
\]
\[
t_m'(\beta_1', \tau', 0) = t_m'(\beta_1', \tau')
\]
(3.20)
and
\[
t_m^0(\beta_1, 0) = t_m(\beta_1, \beta_2, 0, 0) \\
= e^{-2\theta} \frac{d^2}{du^2} + e^{-2\theta} \frac{m^2 - 1}{4u^2} - e^{-\theta} \frac{\beta_1}{u} + \frac{1}{4}.
\]
(3.21)

**Proposition III.3.** Let \((\beta_1', \tau') \in \Omega \times \mathbb{C} \setminus (\mathbb{R}^+ \cup \{0\})\), \(\Omega\) open, bounded, and simply connected in the half-plane \(\text{Re} \beta_1' > 0\). Then, for \(|m| = 0, 1, \ldots:\)

1. \(T_m(\beta_1', \tau')\), \(T_m^0(\beta_1')\) are type-A, real-holomorphic families (in the sense of Kato [16, Sect. VII.1]) of \(m\)-sectorial operators in \((\beta_1', \tau')\) jointly and in \(\beta_1'\), respectively, and thus self-adjoint for \((\tau', \beta_1') \in \mathbb{R}^+ \times \mathbb{R}^+\).

2. \(\sigma_{\text{ess}}(T_m'(\cdot)) = \sigma_{\text{ess}}(T_m^0(\cdot)) = [\frac{1}{4}, +\infty)\) for all \((\beta_1', \tau')\).

3. Given \(\mu_1(m, k) > 0\) there is \(0 < M_1(m, k) < \infty\) such that each eigenvalue \(\lambda(m, k; \beta_1')\) of \(T_m^0(\beta_1')\), \(|m|, k) = 0, 1, \ldots,\) is stable as an eigenvalue \(\lambda' (m, k; \beta_1', \tau')\) of \(T_m'(\beta_1', \tau')\) for \(|\tau'| < M_1, |\arg \tau'| < \pi - \mu_1\).
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(4) Each eigenvalue $\lambda'(\cdot, \beta'_1, \tau')$ is holomorphic in $z(\tau', \beta'_1)$ jointly for $0 < |\tau'| < M$, $|\arg \tau'| < \pi - \mu$, locally in $\beta'_1$, and admits analytic continuation with respect to $\tau'$ to the Riemann-surface sector $\mathcal{D}_1(m, k) = \{ \tau' : 0 < |\tau'| < M(m, k); |\arg \tau'| < \frac{\pi}{2} \pi - \mu \}$ across the negative real axis.

(5) \( \lim_{m \rightarrow 1} \lambda'(m, k; \beta'_1, \tau') = \lambda(m, k; \beta'_1) \) as $|\tau'| \rightarrow 0$ within $\mathcal{D}_1(m, k)$, uniformly with respect to $\beta'_1 \in \Omega$.

Proof. It is well known that the quadratic form

$$t^0_m(f, g) := \left( \left( -\frac{d^2}{du^2} + \frac{m^2 - 1}{4u^2} \right) f, g \right)_{L^2(0, \infty)},$$

$(f, g) \in H^1_0[0, +\infty)$, if $m > 1$, $(f, g) \in H^1(0, \infty)$ and $(f(u), g(u)) = O(u^{1/2})$ as $u \rightarrow 0$ for $m = 0$, is symmetric, closed, and positive. The associated self-adjoint operator on $L^2(0, \infty)$ is $T^0_m$, defined as the action of $-\frac{d^2}{du^2} + \frac{m^2 - 1}{4u^2}$ on $D \equiv \{ H^1_0[0, \infty) \cap H^2(0, \infty), m > 0; H^2(0, +\infty) \}$ with boundary condition $f(u) = O(u^{1/2})$ as $u \downarrow 0, m = 0$. By the Sobolev inequality, the maximal multiplication operator by $u^{-1}$ on $L^2(0, \infty)$ is compact from $D$ to $L^2(0, \infty)$, and the same is true for the maximal multiplication operator by $q(u/m, \beta'_1, \tau', 0)$ in $L^2(0, \infty)$ as long as $|\arg \tau'| < \pi$. Hence by standard results of perturbation theory $T^0_m(\beta'_1)$ and $T'_m(\beta'_1, \tau')$ are closed and $m$-sectorial, and thus self-adjoint for $(\beta'_1, \tau') \in \mathbb{R}^+ \times \mathbb{R}^+$. Furthermore, clearly $\sigma_{\text{ess}}(T^0_m) = [\frac{1}{4}, +\infty)$, and thus by Weyl's theorem, $\sigma_{\text{ess}}(T^0_m(\beta'_1)) = \sigma_{\text{ess}}(T'_m(\beta'_1, \tau')) = \sigma_{\text{ess}}(T^0_m) = [\frac{1}{4}, +\infty)$ for all $(\beta'_1, \tau') \in \Omega \times \{ \tau' : |\arg \tau'| < \pi \}$. Moreover, $D(T^0_m(\beta'_1)) = D(T'_m(\beta'_1, \tau'))$ is $(\beta'_1, \tau')$-independent, and the $L^2$-valued functions $\beta'_1 \mapsto T^0_m(\beta'_1) f, (\beta'_1, \tau') \mapsto T'_m(\beta'_1, \tau') f$ are holomorphic in $\Omega$ and $\Omega \times \{ \tau' : |\arg \tau'| < \pi \}$, respectively, for any $f \in D$. Therefore, the operator families $T^0_m(\beta'_1)$ and $T'_m(\beta'_1, \tau)$ are type-A holomorphic by definition, with the property $(T^0_m(\beta'_1))^* = T^0_m(\beta'_1), (T'_m(\beta'_1, \tau'))^* = T'_m(\beta'_1, \tau')$. This verifies (1) and (2). To see (3), it is enough, by standard arguments of perturbation theory (see, e.g., Simon [27]), to prove that $T'_m(\beta'_1, \tau')$ converges in norm-resolvent sense to $T^0_m(\tau')$ as $|\tau'| \rightarrow 0$, uniformly with respect to $(\beta'_1, |\arg \tau'|) \in \Omega \times [0, \pi - \mu]$. By the uniform $m$-sectoriality, $\| (T'_m(\beta'_1, \tau') - z)^{-1} \| \leq C$ for $z$ negative and $|z|$ suitably large and some $C > 0$ independent of $(\beta'_1, \tau') \in \Omega \times \{ \tau' : |\tau'| < M; |\arg \tau'| \leq \pi - \mu \}$. Since $D(T'_m(\cdot))$ is independent of $\tau'$, we can write

$$\begin{align*}
(T'_m(\beta'_1, \tau') - z)^{-1} - (T^0_m(\beta'_1) - z)^{-1} &= (T'_m(\beta'_1, \tau') - z)^{-1} q(u, m; \beta'_1, \tau', 0)(T^0_m(\beta'_1) - z)^{-1}.
\end{align*}
$$

(3.22)

Now the norm of the right side of (3.22) is majorized by $C \| q(\cdot) C(T^0_m(\cdot) - z)^{-1} \| \leq C' \| q(u, \cdot) \|_{L^\infty(0, +\infty)} \| (T^0_m(\beta'_1) - z)^{-1} \| \leq C^2 \sup_{u \geq 0} \| q(u, m; \beta'_1, \tau', 0) \| \rightarrow 0$ as $|\tau'| \rightarrow 0$ with the stated uniformity in $(\beta'_1, |\arg \tau'|)$. This proves assertion (3). The holomorphy statement of assertion (4) is a well-known consequence of the stability and of the holomorphy of the operator family $T'_m(\beta'_1, \tau')$. 


To see the existence of the analytic continuation we use the complex-scaling technique of Aguilar, Balslev, and Combes (see, e.g., Reed and Simon [15, XIII.10]). The dilatation map

$$(U(\theta) f)(u) = e^{\theta \rho} f(e^{\theta} u), \quad \theta \in \mathbb{R},$$

(3.23)
is unitary on $L^2(0, +\infty)$ and leaves $D$ invariant. The unitary images of $T^0_m(\beta'_1)$ and $T^0_m(\beta'_1, \tau')$ are the operators $T^0_m(\beta'_1, \theta)$ and $T^0_m(\beta'_1, \tau', \theta)$ defined as the action on $D$ of the differential expressions (3.21) and (3.19). Proceeding as in the verification of assertions (1) and (2), we see that $T^0_m(\beta'_1, \theta)$ extends to a type-A, real-holomorphic family of $m$-sectorial operators in $(\beta'_1, \tau', \theta) \in \Omega \times \{ \theta : |\text{Im} \theta| < \pi/2 \}$, and that $T^0_m(\beta'_1, \tau', \theta)$ extends to a type-A, real-holomorphic family of $m$-sectorial operators in $(\beta'_1, \tau', \theta) \in \Omega \times \{ \tau' : |\text{arg} (\tau' e^{\theta})| < \pi \}$. Furthermore, $\sigma_{\text{ess}}(T^0_m(\cdot)) = \sigma_{\text{ess}}(T^0_m(\cdot)) = [e^{-2\lambda \zeta^2 + \frac{1}{4}}, \zeta \in \mathbb{R}]$, and the eigenvalues of both families are independent of $\theta$. The norm-resolvent convergence of assertion (3) holds unchanged also in the present situation provided $|\text{arg} (\tau' e^{\theta})| < \pi - \mu_1$. Therefore, the eigenvalues $\lambda(m, \beta'_1)$ are stable as eigenvalues $\lambda'(m, \beta'_1, \tau')$ of $T^0_m(\beta'_1, \tau', \theta)$ for $|\tau'| < M_1$, $|\text{arg} (\tau' e^{\theta})| < \pi - \mu_1$. Since $|\text{Im} \theta| < \pi/2$, we see that $\lambda'(\cdot, \beta'_1, \tau')$ admits analytic continuation to $|\tau'| < M_1$, $|\text{arg} (\tau')| < \frac{3}{2} \pi - \mu$, a priori many-valued because $\lambda'(\cdot, e^{in} \tau') \neq \lambda'(\cdot, e^{-\pi} \tau')$, $\tau' > 0$, $\beta'_1 \in \mathbb{R}$. In fact, $\lambda'(\cdot, e^{in} \tau')$ is by definition an eigenvalue of $T^0_m(\cdot, \theta)$ for $-\pi/2 < \text{Im} \theta < 0$, while $\lambda'(\cdot, e^{-\pi} \tau')$ is an eigenvalue of $T^0_m(\cdot, \theta)$ for $0 < \text{Im} \theta < \pi/2$. Since $T^0_m(\cdot, \theta) = T^0_m(\cdot, \theta)$, $\text{Im} \lambda'(\cdot, e^{\pi} \tau') = -\text{Im} \lambda'(\cdot, e^{\pi} \tau')$, $\tau' > 0$. This proves (4) and (5).

Proposition III.4. Let $(m, k)$ be fixed, $\beta'_1 \in \Omega$, $|\text{arg} (\tau e^{\theta})| < \pi$. Let $\lambda'(\cdot, \tau')$, $\tau' \in \mathcal{D}_1(\cdot)$ be the eigenvalue of $T^0_m(\cdot, \tau', \theta)$ near the eigenvalue $\lambda(\cdot) \in \sigma_m(\cdot)$ of $T^0_m(\cdot, \theta)$. Then:

1. The Rayleigh–Schrödinger perturbation expansion $\sum_{n=0}^{\infty} A_n(\cdot, \beta'_1)(\tau'/2)^n$, $A_0 = \lambda(\cdot)$, exists and represents a strongly asymptotic expansion (see, e.g., Reed and Simon [15, Sect. XII.4]) for $\lambda'(\cdot, \beta'_1, \tau')$ as $|\tau'| \to 0$, uniformly in $(\beta'_1, |\text{arg} \tau'| \in \mathcal{D}_1 \times [0, \frac{3}{2} \pi - \mu_1]$, i.e., given $\mu_1 > 0$ there is $B(\mu_1) > 0$ such that

$$|R_N(\cdot, \tau')| \equiv |\lambda'(\cdot, \tau') - \sum_{n=0}^{N-1} A_n(\cdot)(\tau'/2)^n| \leq B(\mu_1) N!|\tau'/2|^N,$$

(3.24)

$$(\tau', \beta'_1) \in \mathcal{D}_1(\cdot) \times \mathcal{D}_1(\cdot), \quad N = 1, 2, \ldots.$$

2. The perturbation expansion given above is Borel summable to $\lambda'(\cdot, \beta'_1, \tau')$ for $\tau' \in \mathcal{D}_1(\cdot)$, uniformly in $\beta'_1 \in \mathcal{D}_1$.

3. $A_n(m, k; \beta'_1) = (-1)^n B_n(m, k; \beta'_1)$, $n \in \mathbb{N}$.

Proof. By the Watson–Nevanlinna theorem (for details see Reed and Simon [15, Sect. XII.5] and Sokal [27]), given Proposition III.3(4), (5), assertion (2) is a consequence of (1). We prove (1) by standard arguments of perturbation theory.
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(see, e.g., Reed and Simon [15, Sects. XII.2–4]). Let $d = d(m, k; \beta'_1)$ be the isolation distance of the eigenvalue $\lambda'(\cdot, \beta'_1)$, $0 < \nu < \frac{1}{2} d$, and let $\mathcal{F}_\nu = \{ z \in \mathbb{C} : |z - \lambda(\cdot)| = \nu \}$.

Denote by $R'_m(z, \beta'_1, \tau', \theta)$, $R^0_m(z, \beta'_1, \theta)$ the resolvents of $T'_m(\cdot)$, $T^0_m(\cdot)$, respectively. By the norm-resolvent convergence of Proposition III.3 there is a constant $C > 0$ independent of $(\tau', \beta'_1, \theta)$ as long as $\beta'_1 \in \Omega$, $|\arg(e^{\theta} \tau')| \leq \pi - \mu_1$, $|\tau'| < M_1$, such that

$$\sup_{z \in \mathcal{F}_\nu} \| R'_m(z, \beta'_1, \tau', \theta) \| \leq C, \quad (3.25)$$

and furthermore

$$\| P'_m(\beta'_1, \tau', \theta) - P^0_m(\beta'_1, \theta) \| \to 0 \quad \text{as} \quad |\tau'| \to 0 \quad (3.26)$$

uniformly in $\beta'_1 \in \Omega$ and $(|\arg \tau'|, \theta), |\arg(\tau' e^{\theta})| \leq \pi - \mu_1$. Here the strong Riemann integrals

$$P'_m(\beta'_1, \tau', \theta) = (2\pi i)^{-1} \int_{\mathcal{F}_\nu} R'_m(z, \beta'_1, \tau', \theta) \, dz \quad (3.27)$$

and

$$P^0_m(\beta'_1, \theta) = (2\pi i)^{-1} \int_{\mathcal{F}_\nu} R^0_m(z, \beta'_1, \theta) \, dz \quad (3.28)$$

are the projection operators on the one-dimensional eigenspaces of $\lambda'(\cdot, \beta'_1, \tau')$ and $\lambda(\cdot, \beta'_1)$. If $\phi = \phi(\cdot, \beta'_1, \theta)$ denotes the eigenvector corresponding to $\lambda(\cdot, \beta'_1)$, we have

$$\lambda'(\cdot, \beta'_1, \tau') = \frac{\langle P'_m(\beta'_1, \tau', \theta) \phi, T'_m(\beta'_1, \tau', \theta) P'_m(\beta'_1, \tau', \theta) \phi \rangle}{\langle P'_m(\beta'_1, \tau', \theta) \phi, P'_m(\beta'_1, \tau', \theta) \phi \rangle}. \quad (3.29)$$

Recall now that the Rayleigh–Schrödinger expansion is generated by inserting the geometric expansion of the resolvent in powers of the perturbation, as represented by formulae (2.28), (3.11) with $e^{\theta} u$ in place of $u$, collecting all the terms having the same power of $\tau'$, and performing the integration by the residue method. We also recall that by standard complex-scaling arguments the resulting coefficients $A'_m(\cdot)$ are independent of $\theta$. Now, by standard arguments of singular perturbation theory (see, e.g., Reed and Simon [15, Sects. XII.3, 4] and in particular Morgan and Simon [3] for a specific application to the present case in the non-separated formalism), to see (3.24) it is enough to prove that there are $\sigma(v) > 0$, $C(v) > 0$ independent of $(\beta'_1, \tau', \theta)$, $\beta'_1 \in \Omega$, $|\arg(\tau' e^{\theta})| \leq \pi - \mu_1$, such that

$$\sum_{k_1 + \cdots + k_l = N} \sup_{z \in \mathcal{F}_\nu} \| R^0_m(z, \beta'_1, \theta) F_{k_1}(e^{\theta} u, \beta'_1) R^0_m(\cdot) \cdots F_{k_l}(\cdot) \phi(\beta'_1, \theta) \| \leq C \sigma^N N! \quad (3.30)$$
and since the number of terms in this sum is dominated by $4^N$ we need only prove the bound for each term separately. To this end, we first recall that under the present conditions it is well known that there are $\delta_1 > 0$ and $C_1 > 0$ independent of $(\beta'_1, \theta) \in \bar{Q} \times \{\theta : \text{Im } \theta \leq \pi/2 - \varepsilon, \varepsilon > 0\}$ such that $\|e^{\delta_1 u} \phi(\cdot, \beta'_1, \theta)\| \leq C'_1$. Furthermore, there is $C_2 > 0$ independent of $(\beta'_1, \theta)$ as above such that

$$\sup_{0 \leq \delta \leq \delta_0, z \in I} \| e^{\delta u} R^0_m(z, \beta'_1, \theta) e^{-\delta u} \| \leq C_2. \quad (3.31)$$

To see this, we apply a well-known argument (see, e.g., Hunziker and Pillet [28]): for $f \in D$, we compute

$$e^{\delta u} T^0_m(\beta'_1, \theta) e^{-\delta u} f = T^0_m(\beta'_1, \theta) f - \delta^2 u + 2e^{-\theta} \delta pf, \quad p = -i \frac{d}{du}.$$ 

Now $p$ is obviously $T^0_m(\cdot)$-bounded with relative bound zero, uniformly in $(\beta'_1, \theta) \in \bar{Q} \times \{\theta : \text{Im } \theta \leq \pi/2 - \varepsilon, \varepsilon > 0\}$. Hence (3.13) follows by a standard argument, described, e.g., in Morgan and Simon [3], for $\delta_1$, and hence $\delta$, small enough. Now the rest of the argument goes exactly as in Morgan and Simon [3]. We write

$$R^0_m(z, \beta'_1, \theta) F_{k_1}(e^{\delta u} \beta'_1) \cdots F_{k_l}(\cdot) R^0_m(\cdot) \phi(\cdot, \beta'_1, \theta)$$

$$= \bar{Q}_0 \bar{P}_1 \bar{Q}_1 \cdots \bar{Q} e^{\delta u} \phi(\cdot, \beta'_1, \theta) \quad (3.32)$$

and

$$\bar{P}_i = F_{k_i}(\cdot) e^{-k_i \delta u / N}, \quad \bar{Q}_i = e^{j_i \delta u / N} R^0_m(\cdot) e^{-j_i \delta u / N},$$

$$j_i = \sum_{s=1}^i k_s. \quad (3.33)$$

Now $\|\bar{Q}\| \leq C_2$, and $\|\bar{P}_i\| = \|F_{k_i}(\cdot) e^{-k_i \delta u / N}\|_{L^\infty} \leq N^{k_i - 1} C_3$ for some $C_3 > 0$ independent of $(\beta'_1, \theta)$ as above. Thus each term of (3.30) is majorized by $C_3^N C^{N+1} N^N \leq C\sigma(N!)$ for some $\sigma(v) > 0$, whence (3.30). Therefore (1), and hence (2), is proved. To see (3), it is enough to remark that $F'(u, \beta'_1, \tau e^{-i\tau}) = G(u, \beta'_1, \tau)$, $\tau > 0$, while the unperturbed operator is the same in both cases and the perturbation expansion is independent of $\theta$. $\square$

As an immediate consequence of this proposition we have:

**Corollary III.5.** The Rayleigh–Schrödinger perturbation expansion $\sum_{n=0}^{\infty} B_n(m, k; \beta_2) (\tau/2)^n$ for the eigenvalue $\mu(m, k; \beta_2, \tau)$ of $S_m(\beta_2, \tau)$ is Borel summable not to $\mu_\pm(m, k; \beta_2, \tau)$ but to $\lambda'(m, k; \beta_2, e^{-i\tau})$, $\tau > 0$.

The second step in proving Theorem III.2 is represented by the unraveling of the first separation-constant eigenvalues.
Proposition III.6. Let \(|m| = 0, 1, \ldots, \beta'_1 \in \Omega, \; |\arg(\tau'e^\theta)| < \pi.\) Denote by \(\sigma'(m, \tau', \theta)\) and \(\sigma_0(m, \theta)\) the charge spectra of \(T'_m(\cdot)\) and \(T'_0(\cdot)\), respectively, i.e., the sets \(\{\beta'_1 \in \Omega: T'_m(\beta'_1, \tau', \theta) \text{ has the eigenvalue } 0\}\) and \(\{\beta'_1 \in \Omega: T'_0(\beta'_1, \theta) \text{ has the eigenvalue } 0\}\). Then:

1. \(\sigma'(m, \tau', \theta) = \sigma'(m, \tau', 0) = \sigma'(m, \tau'); \; \sigma_0(m, \theta) = \sigma_0(m, 0) = \sigma_0(m),\) i.e., the charge spectra are independent of \(\theta\).

2. For any fixed \((|m|, k) = 0, 1, \ldots,\) and any \(\mu_2(m, k) > 0,\) there is \(0 < M_2(m, k) < +\infty\) such that the condition \(\lambda'(m, k; \beta'_1, \tau) = 0\) implicitly defines one and only one isolated eigenvalue in \(\sigma'(m, \tau')\) as a function \(\tau' \mapsto \beta'_1(m, k, \tau')\), holomorphic for \(0 < |\tau'| < M_2, \; |\arg \tau'| < \pi,\) which admits analytic continuation to the Riemann-surface sector \(\mathcal{D}_2(m, k) = \{\tau': 0 < |\tau'| < M_2; \; |\arg \tau'| < \frac{3}{2} \pi - \mu_1\}\) across the negative real axis, and is such that \(\beta'_1(m, k; \tau') \to \beta(m, k) = k + \frac{1}{2}(|m| + 1)\) as \(|\tau'| \to 0, \; \tau' \in \mathcal{D}_2(m, k)\).

3. The function \(\tau' \mapsto \beta'_1(m, k; \tau')\) admits an asymptotic expansion to all orders,

\[
\beta'_1(m, k; \tau') \sim \sum_{n = 0}^\infty L'_n(m, k)(\tau'/2)^n, \quad L'_0(m, k) = \beta(m, k) \quad (3.34)
\]

as \(\tau' \to 0\) within \(\mathcal{D}_2(m, k)\). The coefficients \(L'_n(m, k)\) can be directly computed through Rayleigh–Schrödinger perturbation theory.

4. The asymptotic expansion (3.34) is Borel summable to \(\beta'_1(m, k; \tau')\) in \(\mathcal{D}_2(m, k)\).

Proof. Assertion (1) is an immediate consequence of dilatation analyticity. To see the subsequent ones, first recall that \(\lambda(m, k; \beta'_1) = 0\) if and only if \(\beta'_1 = \beta(m, k)\), i.e., \(\sigma_0(m) = \bigcup_{n = 0}^\infty B(m, k)\). The corresponding eigenfunctions \(\phi(m, k, \theta) = \phi(m, k, e^\theta u)\) are the Laguerre functions of argument \(e^\theta u\). Consider the eigenvalue \(\lambda'(m, k; \beta'_1, \tau')\) existing near \(\lambda(m, k, \beta'_1)\) for \(\beta'_1 \in \Omega\) and \(\tau' \in \mathcal{D}_1(m, k)\). By Proposition III.4, uniformly with respect to \(\beta'_1 \in \Omega,
\[
\lambda'(m, k; \beta'_1, \tau') = \lambda(m, k; \beta'_1) + O(m, k; \tau'/2) \quad (3.35)
\]

as \(|\tau'| \to 0\) within \(\mathcal{D}_1(m, k)\). Furthermore (see Buchholz [24]), \(\lambda(m, k; \beta'_1) = \frac{1}{4} - (\beta'_1)^2/4[k + \frac{1}{2}(|m| + 1)]^2\) and thus \(\beta(m, k) \in \Omega, \; (\partial \lambda/\partial \beta'_1)(m, k; \beta'_1)|_{\beta'_1 = \beta(m, k)} \neq 0\). Hence (3.35) implies, by continuity, that

\[
\frac{\partial y}{\partial \beta'_1}(m, k; \beta'_1, \tau') \neq 0
\]

for \(|\beta'_1 - \beta(m, k)|\) suitably small and \(\tau' \in \mathcal{D}_1(m, k), \; |\tau'|\) suitably small. Since \(\lambda'(m, k; \beta(m, k), \tau') \to \lambda(m, k; \beta(m, k)) = 0\) as \(\tau' \to 0\) within \(\mathcal{D}_1(m, k)\), assertion (2) is a direct consequence of the analytic implicit-function theorem (see, e.g., Gallavotti [29, Appendix G]). Furthermore, the analytic implicit-function theorem also implies that \(\beta'_1(m, k; \tau')\) has finite derivatives of all orders as \(\tau' \in \mathcal{D}_2(m, k) \to 0\). To
compute these derivatives, viz., the coefficients \( L_n^\ast(m, k) \), notice that \( \beta(m, k) \) satisfies

the ordinary differential equation \( e^u \phi_n^0(0, \theta) \phi(m, k; e^u) = \beta(m, k) \phi(m, k; e^u) \).

Hence if we consider the ODE eigenvalue problem

\[
[e^u \phi_n^0(0, \theta) + e^u G(\beta, \phi, \beta') \phi(m, k; e^u, \tau')] \phi'(m, k; e^u, \tau') = \beta_1 \phi'(m, k; e^u, \tau')
\]

on \( L^2(\mathbb{R}^+; d\chi) \), \( d\chi = u^{-1} du \), with boundary condition \( \phi'(m, \cdot) = O(u^{1/2 + |m|/2}) \) as \( u \downarrow 0 \), we generate the coefficients \( L_n^\ast(m, k) \) recursively through Rayleigh-Schrödinger perturbation theory. Note that this formal procedure is justified because \( \|\phi'(m, k; \tau', \theta) u^{-1}\| \) is bounded independently of \( |\tau'| \), and

\[
[e^u (T_n^0(0, \theta) - z)]^{-1} e^u F'(m, e^u, \beta_1') = [T_n^0(0, \theta) - z]^{-1} F'(m, e^u, \beta_1').
\]

Finally, assertion (4) follows by Proposition A.1.

**Corollary III.7.** Let \((|m|, k) = 0, 1, ... \) be fixed, and let \( \tau > 0 \). Then the separation-constant eigenvalue doublet \( \beta_\pm^\ast(m, k, \tau) \) implicitly defined by \( \mu_\pm(m, k, \beta_2, \tau) = 0 \) admits an asymptotic Rayleigh–Schrödinger perturbation expansion

\[
\beta_\pm^\ast(m, k; \tau) \sim \sum_{n=0}^{\infty} L_n(m, k)(\tau/2)^n, \quad L_0 = \beta(m, k),
\]

which is Borel summable not to \( \beta_\pm^\ast(m, k) \) but to \( \beta_1'(m, k, e^{-in} \tau) \).

**Proof.** The expansion (3.37) can be generated as in Proposition III.6(3) considering this time the ODE eigenvalue problem \( [v \phi_n^0(0) + v G(\beta_2, \tau, v)] \psi(m, k; \tau, v) = \beta_2(m, k; \tau, v) \) (see Proposition II.3, (2.29)–(2.30), (3.11)) on \( L^2(\mathbb{R}^+, d\chi) \) with boundary condition \( \psi(m, \cdot, v) = O(v^{1/2 + |m|/2}) \) as \( v \downarrow 0 \). Here, as usual,

\[
e_\ast^0(\beta_2) = \frac{d^2}{dv^2} \frac{\beta_2}{v} + \frac{m^2 - 1}{4v^2} + \frac{1}{4}.
\]

By Corollary III.5, we have \( L_n(m, k) = (-1)^n L_n'(m, k) \), with \( L_n'(m, k) \) as in (3.34). Therefore the assertion is implied by (4) of Proposition III.6.

The analysis of the operator family \( T_m(\beta_1, \beta_2, \tau) \) is now straightforward. By exactly the same arguments as in Propositions III.3 and III.4, we obtain:

**Proposition III.8.** Let \((\beta_1, \beta_2, \tau) \in \Omega \times \Omega \times \{\tau: |\arg \tau| < \pi\}, \Omega \text{ as in Proposition III.3. Let } T_m(\beta_1, \beta_2, \tau) \text{ be the operator family on } L^2(0, + \infty) \text{ defined by the differential expression } t_m(\beta_1, \beta_2, \tau) \text{ on } D, D \text{ as in Proposition III.3. Then:}

(1) \((\beta_1, \beta_2, \tau) \rightarrow T_m(\beta_1, \beta_2, \tau), |m| = 0, 1, ..., \) is a type-\(\Lambda\), real-holomorphic family of \( m \)-sectorial operators in \((\beta_1, \beta_2, \tau) \in \Omega \times \Omega \times \{\tau: |\arg \tau| < \pi\}, \) and thus self-adjoint for \((\beta_1, \beta_2, \tau) \in \mathbb{R}\).
(2) \( \sigma_{\text{ess}}(T_m(\cdot)) = \left[ \frac{1}{4}, +\infty \right) \) for any \((\beta_1, \beta_2, \tau) \in \Omega \times \Omega \times \{ \tau : |\arg \tau| < \pi \} \).

(3) Given \( \mu_3(m, j) > 0 \) there is \( \lambda_3(m, j) > 0 \) such that each eigenvalue \( \lambda(m, j, \beta_1) \) of \( T_m(\beta_1) \) is stable as an eigenvalue \( \lambda(m, j; \beta_1, \beta_2, \tau) \) for \( |\tau| < \lambda_3 \), \( |\arg \tau| < \pi \); the function \( \lambda(m, j; \beta_1, \beta_2, \tau) \) is holomorphic in \((\beta_1, \beta_2, \tau)\), jointly for \( 0 < |\tau| < \lambda_3 \), \( |\arg \tau| < \pi \), and locally in \((\beta_1, \beta_2) \in \Omega \times \Omega \), and admits analytic continuation with respect to \( \tau \) to the Riemann-surface sector \( \mathcal{D}_2(m, j) = \{ \tau : 0 < |\tau| < \lambda_3(m, j) ; |\arg \tau| < \frac{3}{2} \pi - \mu_3 \} \) across the negative real axis. Furthermore, \( \lim \lambda(m, j; \beta_1, \beta_2, \tau) = \lambda(m, j, \beta_1) \) as \( \tau \to 0 \) within \( \mathcal{D}_2(m, j) \) uniformly in \((\beta_1, \beta_2) \in \Omega \times \Omega \).

(4) The Rayleigh–Schrödinger perturbation expansion

\[
\sum_{n=0}^{\infty} A_n(m, j; \beta_1, \beta_2)(\tau/2)^n, \quad A_0 = \lambda(m, j, \beta_1),
\]

exists, represents a strong asymptotic expansion for \( \lambda(m, j; \beta_1, \beta_2, \tau) \) as \( \tau \to 0 \), \( \tau \in \mathcal{D}_2(m, j) \), uniformly with respect to \((\beta_1, \beta_2) \in \Omega \times \Omega \), and is Borel summable to \( \lambda(m, i; \beta_1, \beta_2, \tau) \) in \( \mathcal{D}_2(m, j) \), uniformly in \((\beta_1, \beta_2) \) as above.

These results, together with Proposition III.6, Proposition A.1, and Corollary A.2, immediately imply:

**Corollary III.9.** For \( \tau \in \mathcal{D}_2(m, j) \), consider the eigenvalue \( \lambda(m, j; \beta_1, \beta_2, \tau) \) and the \( \beta_1 \)-separation-constant eigenvalue \( \tau \to \beta'_1(m, k; \tau') \) of Proposition III.6, \( \tau' \in \mathcal{D}_2(m, k) \), \( (|m|, j, k) = 0, 1, \ldots \). Then:

(1) The function \( \tau \to \lambda(m, j; \beta_1, \beta'_1(m, k, \tau e^{-im}), \tau) \) is holomorphic in \((\beta_1, \tau)\) for \( 0 < |\tau| < M_4(m, j, k) = \min (M_2(\cdot), M_3(\cdot)) \), \( 0 < \arg \tau < \pi \), locally in \( \beta_1 \in \Omega \). Furthermore, \( \lambda(\cdot, \beta'_1(\cdot, \tau e^{-im}), \tau) \) admits analytic continuation to the Riemann-surface sector \( \mathcal{D}_4(m, j, k) = \{ \tau : 0 < |\tau| < M_4(\cdot) ; -\pi/2 + \mu_4(\cdot) < \arg \tau < \frac{3}{2} \pi - \mu_4(\cdot) \} \), \( \mu_4(m, j, k) = \max (\mu_1(\cdot), \mu_2(\cdot)) \), across the real axis, with \( \lim \lambda(m, j; \beta_1, \beta'_1(m, k, \tau e^{-im}), \tau) = \lambda(m, j, \beta_1) \) as \( \tau \to 0 \) within \( \mathcal{D}_4(m, j, k) \), uniformly with respect to \( \beta_1 \in \Omega \).

(2) The Rayleigh–Schrödinger perturbation expansion for \( \lambda(m, j; \beta_1, \beta'_1(m, k, \tau e^{-im}), \tau) \), viz.,

\[
\lambda(m, j; \beta_1, \beta'_1(m, k, \tau e^{-im}), \tau) \sim \sum_{n=0}^{\infty} A_n(m, j, k; \beta_1)(\tau/2)^n,
\]

exists, is strongly asymptotic to \( \lambda(\cdot, \tau) \) as \( \tau \to 0 \) within \( \mathcal{D}_4(\cdot) \), uniform in \( \beta_1 \in \Omega \), and is Borel summable to \( \lambda(m, j; \beta_1, \beta'_1(m, k, \tau e^{-im}), \tau) \) in \( \mathcal{D}_4(m, j, k) \), uniformly with respect to \( \beta_1 \in \Omega \).

**Remark.** Equation (3.38) is also the perturbation expansion of \( \lambda(m, j; \beta_1, \beta_1^+ (m, k; \tau), \tau) \), because \( \beta_1^+ (m, k, \tau) \) and \( \beta'_1(m, k, e^{-im} \tau) \) have the same perturbation expansion.

The \( \beta_1 \) spectrum is now determined as follows:
Proposition III.10. For \(|m|, j, k) = 0, 1, \ldots\), consider the eigenvalue \((m, j; \beta_1, \beta_1'(m, k; \tau e^{-in}), \tau)\) of \(T_m(\beta_1, \beta_1'(e^{-in}), \tau)\). Then:

1. The condition that
\[
\lambda(m, j; \beta_1, \beta_1'(m, k; \tau e^{-in}), \tau) = 0
\]
implicitly defines a function \(\tau \mapsto \beta_1(m, j, k)\), which is holomorphic for \(0 < |\tau| < M_4(m, j, k)\), \(0 < \arg \tau < \pi\), admits analytic continuation to the Riemann-surface sector \(D_4(m, j, k)\), and is such that \(\lim \beta_2(m, j, k; \tau) = \beta(m, i) = i + \frac{1}{2}(|m| + 1)\) as \(\tau \to 0\) within \(D_4(m, j, k)\).

2. The implicit function \(\tau \mapsto \beta_1(m, j, k; \tau)\) admits the Rayleigh-Schrödinger perturbation expansion
\[
\beta_1(m, j, k) \sim \sum_{n=0}^{\infty} L_n(m, j, k)(\tau/2)^n, \quad L_0 = \beta(m, i)
\]
as a strongly asymptotic expansion as \(\tau \to 0, \tau \in D_4(m, j, k)\). The expansion (3.40) is Borel summable to \(\beta_1(m, j, k; \tau)\) for \(\tau \in D_4(m, j, k)\).

Proof. (1) Since \(\lambda(m, j, \beta(m, j)) = 0\), proceeding as in Proposition III.6 we have to prove only that
\[
\frac{\partial \lambda}{\partial \beta_1}(m, j; \beta_1, \beta_1'(m, k, \tau e^{-in}, \tau)) \neq 0
\]
for \(\beta_1\) in a neighborhood of \(\beta(m, j)\) and \(\tau \in D_4(m, j, k)\) with \(M_4\) suitably small. In turn, by Proposition III.8(4) it is enough to check that
\[
\frac{\partial}{\partial \beta_1} A_0(m, j, k; \beta_1) \bigg|_{\beta = \beta(m, j)} \neq 0,
\]
which is true because \(A_0(m, j, k; \beta_1) = \lambda(m, j, \beta_1) = \frac{1}{4} - \beta_1^2/4[j + \frac{1}{2}(|m| + 1)]^2\). Assertion (2) is again proved as in Proposition III.6(3) and Proposition A.1, given Proposition 3.9(1) and (2). We note that by the remark after Proposition III.9 the functions \(\tau \mapsto \beta_1(m, j, k; \tau)\) and \(\tau \mapsto \beta_1(m, j; \beta_2(m, k; \tau), \tau)\) have the same perturbation expansion (3.40).

Proof of Theorem III.2. Setting \(M(m, j, k) = \min\{M_1(\cdot), \ldots, M_4(\cdot)\}, \mu(m, j, k) = \max\{\mu_1(\cdot), \ldots, \mu_4(\cdot)\}\), assertion (1) is proved in Proposition III.6, and assertion (2) in Proposition III.10. Assertion (3) follows from (1), (2), and the analytic local-invertibility theorem, because
\[
\frac{\partial}{\partial \tau} [\tau \gamma_1(m, j, k; \tau)^{-1}] = (j + k + |m| + 1)^{-1} + O(m, j, k; \rho) \quad \text{as} \ \tau \to 0
\]
within \( \mathcal{D}(m, j, k) \). Finally, note that by Proposition III.4(3), Corollaries III.5, III.7, and III.9, and Proposition III.10, and the analytic local-invertibility theorem, the function \( \rho \to -\frac{1}{2} [\gamma_1(m, j, k; \Gamma_1(m, j, k; \rho))]^{-2} \) admits an asymptotic expansion to all orders as \( \rho \to 0 \) within \( \mathcal{D}(m, j, k) \). Hence assertions (4) and (5) are direct consequences of Corollary III.7, Proposition III.10(2), Propositions A.1 and A.2, and Reed and Simon [15, Problem XII.26].

IV. Imaginary Parts, Asymptotics, and the Formula of Brézin and Zinn-Justin

As stated in the first section, our program now is to relate the Borel sum \( E_1(m, i, k; \rho) \) of the 1/R expansion to the fundamental quantities of the problem, viz., the eigenvalue gap and the asymptotics of the coefficients of the 1/R series itself. In this section, the quantum numbers \( m, j, \) and \( k \) are fixed and may have any allowed value. Although eigenvalues, expansion coefficients, wavefunctions, error estimates, etc., all depend on these numbers, to avoid notational complexity that dependence will be indicated only where necessary. Since the coefficients of the 1/R expansion are real, \( \text{Im} \ E_1 \) must have zero asymptotic expansion as \( \rho \to 0 \). In fact, the asymptotic behavior of \( \text{Im} \ E_1 \) is determined to leading exponential order by the following statement.

**Theorem IV.1.** Let \( E(m, j, k; \rho) \) be the Borel sum of the 1/R expansion near the eigenvalue \( E(m, j, k) = -\frac{1}{2} (|m| + j + k)^{-2} \) of \( -\frac{1}{2} \Delta - |x|^{-1} \) of magnetic quantum number \( m \) and parabolic quantum numbers \( (j, k), (|m|, j, k) = 0, 1, ..., \) and let \( n = |m| + j + k + 1 \) be the principal quantum number. Then, as \( |\rho| \downarrow 0, \rho \in \mathbb{R} \),

\[
\text{Im} \ E_1(m, j, k; \rho) = -\pi C(m, j, k) \left( \frac{2}{n\rho} \right)^{2|m| + 4k + 2} \times e^{-2|m| n}(1 + O(n, j, k; \rho^{1/2}))
\]

with

\[
C(m, j, k) = n^{-3} [\Gamma(k + |m|)!]^{-2} e^{-2n}.
\]

Here, and everywhere else, \( O(m, j, k, \rho^{1/2}) \) means order \( \rho^{1/2} \) as \( \rho \to 0 \) with coefficients depending on \( (m, j, k) \). This theorem will be proved in this section by adapting the ODE techniques of Harrell and Simon [6], which are in essence rigorously justified JWKB estimates. Before turning to that analysis, we note that the asymptotics of the 1/R expansion and the formula of Brézin and Zinn-Justin are simple consequences of Theorems IV.1 and III.2 along with the rigorously known gap estimates of Harrell [13].

**Corollary IV.2.** Let \( E_N(m, j, k) \) be the \( N \)th coefficient of the 1/R expansion near the eigenvalue \( E(m, j, k) \) of \( H_0 \). Then:
(1) As \( N \to \infty \),

\[
E_N(m, j, k) = C(m, j, k) n^{N2^{-N}(N + 4k + 2m + 1)} (1 + O(m, j, k; N^{-1/2})) \\
= -e^{-2nN^{-3}[k!(|m| + k)!]^{-2}2^{-N}(N + 4k + 2m + 1)} (1 + O(m, j, k; N^{-1/2})).
\]  

(4.3)


(2) Let \( \rho > 0 \), and \( \Delta E(m, j, k; \rho) \) be the gap between the two eigenvalues in the doublet near \( E(m, j, k) \) as \( \rho \downarrow 0 \). Then, as \( \rho \downarrow 0 \),

\[-\text{Im } E_1(m, j, k; \rho) = \pi n^3 (\Delta E(m, j, k; \rho))^2 (1 + O(m, j, k; \rho)). \]  

(4.4)

**Remark.** Equation (4.4) is the formula of Brézin and Zinn-Justin, rewritten in the language of the Borel sum. Formula (4.6) below shows that the asymptotic behavior of \( E_N \) is controlled by the eigenvalue gap as well, which was the numerical discovery of Brézin and Zinn-Justin [5].

**Proof.** (1) We use a standard approximate dispersion relation argument which goes back to Simon's paper on the anharmonic oscillator [27]. By Theorem III.2(4), the function \( \rho \mapsto E_1(m, j, k; \rho) \) is holomorphic for \( 0 < |\rho| < M \), \( 0 < \arg \rho < \pi \), and analytic up to the real boundary of this half-circle. If \( \Gamma_\epsilon \) denotes the half-circle \( |z| = \epsilon < M \), \( 0 \leq \arg z \leq \pi \), by Cauchy's theorem,

\[
E_1(m, j, k; \rho) = \frac{1}{2\pi i} \int_{\Gamma_\epsilon} \frac{E_1(m, j, k; z)}{z - \rho} dz.
\]  

(4.5)

Therefore, by Taylor's theorem and the reality of the perturbation coefficients,

\[
E_N(m, j, k) = \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} z^{-N-1} \text{Im } E_1(m, j, k; z) \, dz + O(\epsilon^{-N}),
\]  

(4.6)

and hence (4.1) yields (4.3). Furthermore, assertion (2) is an immediate consequence of (4.1), (4.2), and the known estimate [13]

\[
\Delta E(m, j, k; \rho) = e^{-nN^{-3}[k!(|k + |m|)!]}^{-1} \left( \frac{2}{\mp^2} \right)^{|m| + 2k + 1} e^{-1/\rho^2} (1 + O(\cdot, \rho^{-1/2})).
\]  

(4.7)

To prove Theorem IV.1 it is necessary to estimate the imaginary parts first of \( \beta'_i(\cdot, \tau e^{-i\tau}) \) and then of \( \beta_i(\cdot, \beta_i'(\cdot, e^{-i\tau}), \tau) \), \( \tau \in \mathbb{R} \). As already mentioned, we will make use of the JWKB technique of Harrell [30] and Harrell and Simon [6]. We note in passing that a more sophisticated (but so far not rigorously justified) approach based on the Langer–Cherry refinement of the JWKB method [31] makes the computation of all exponential corrections possible. This is the content of the second paper announced in [14].
The first preliminary result is as follows:

**Proposition IV.3.** For $\tau > 0$, $\Im \theta > 0$, let

$$q'(m; \beta'_1, \tau, e^\theta u) = \frac{1}{4} - e^{-\theta} \beta'_1 u^{-1} - (2r - e^\theta u)^{-1} \beta'_1$$

$$+ \frac{m^2 - 1}{4} [(2r - e^\theta u)^{-2} + 2e^{-\theta} u^{-1} (2r - e^\theta u)^{-1}]$$

(4.8)

and

$$\beta'_1(m, k; \tau e^{-i\pi}) = \beta'_1(m, k; \tau) = \beta'_1(\cdot, \tau).$$

(4.9)

Let $t_2 = t_2(m, k, \tau)$ be the greatest solution in $0 \leq u \leq 2r$ of $q'(m, k\beta(m, k), \tau, u) = 0$, and let $\phi'_1(\cdot, \tau, e^\theta u)$ denote once again the eigenvector corresponding to $\beta'_1(\cdot, \tau)$ in $\sigma'(m, \tau)$. Then:

1. $\lim_{\Im \theta \to 10} \phi'_1(\cdot, \tau, e^\theta u) = \phi'_1(\cdot, \tau, u)$ exists, uniformly in $0 \leq u \leq t_2$.
2. For $0 < a \leq t_2$,

$$\Im \beta'_1(\cdot, \tau) = \frac{\phi'_1(\cdot, \tau, u) \frac{d}{du} \phi'_1(\cdot, \tau, u)|_{u=a} - \phi'_1(\cdot, \tau, u) \frac{d}{du} \phi'_1(\cdot, \tau, u)|_{u=a}}{2i \int_0^a |\phi'_1(\cdot, \tau, u)|^2 (u^{-1} + (2r - u)^{-1}) du}.$$  

(4.10)

**Proof:** By Propositions III.5 and III.6, $\phi'_1$ is the solution in $L^2(0, +\infty)$ of the ODE

$$\left(-e^{-2\theta} \frac{d^2}{du^2} + q'(m, \beta'_1(\cdot, \tau), \tau, e^\theta u)\right) \phi'_1(\cdot, \tau, e^\theta u) = 0$$

(4.11)

for $0 < \Im \theta < \pi/2$. It is well known from standard techniques of asymptotic integration (see, e.g., Hille [32], Olver [33]) that the subdominant solution of (4.11) as $|u| \to \infty$, $u \in \mathbb{C}$, is unique up to constants as long as $|\arg(e^\theta u)| < \pi/2$. Therefore, we can replace the condition $\phi'_1(\cdot, u) \in L^2(0, +\infty)$ by the condition $\phi'_1(\cdot, u) \in L^2(C, d|u|)$, where $C$ is any contour in the complex half-plane $u \in \mathbb{C}$, $\Re u \geq 0$, lying above the singularity at $u = 2re^{-\theta}$. For example, $C = C_1 \cup C_2$; $C_1 = \{u \in \mathbb{C}: \Im u = 0, \ 0 \leq \Re u \leq 2(r - \bar{r}), \ \Re u \geq 2(r + \bar{r})\};$ $C_2 = \{u \in \mathbb{C}: |u - 2r| = 2\bar{r}: \Im u > 0\}$ for some fixed $\bar{r}(m, k) > 0$. Since the regular, subdominant solution of (4.11) is continuous at $\Im \theta = 0$ uniformly with respect to $u \in C$, and the eigenvalues are independent of $\theta$, we may henceforth assume $\Im \theta = 0$. The point $2(r - \bar{r})$ can be taken as the greatest solution $t_2(m, k)$ in $(0, 2r)$ of $q'(m, \beta'_1(\cdot, \tau), \tau, u) = 0$ (the "large
turning point\(^\dagger\)). Formula (4.10) then follows by a standard partial integration argument. In particular

\[
\text{Im } \beta'_t(\cdot, \tau) = \frac{\phi'_1(\cdot, t_2) \frac{d}{du} \phi'_1(\cdot, u) \bigg|_{u = t_2} - \phi'_1(\cdot, t_2) \frac{d}{du} \phi'_1(\cdot, u) \bigg|_{u = t_2}}{2i \int_0^\beta |\phi'_1(\cdot, u)|^2 (u^{-1} + (2r - u)^{-1}) \, du}. \tag{4.12}
\]

Equation (4.12) is the standard formula for estimating imaginary parts, and in order to evaluate it we shall exhibit a patched-together comparison function \(\chi(m, k, \tau, u) = \chi(\cdot, \tau, u)\) such that

\[
\phi'_1(\cdot, \tau, u) = \chi(\cdot, \tau, u)(1 + \varepsilon(\cdot, \tau, u)), \tag{4.13}
\]

where \(|\varepsilon(\cdot, \tau, u)| + |(de/du)(\cdot, \tau, u)| = O(\tau^\alpha)\) for some \(\alpha = \alpha(m, k) > 0, 0 \leq u \leq t_2\).

Since the subsequent arguments are essentially adaptations to the present case of those of Harrell [30] and Harrell and Simon [6], we shall be somewhat sketchy. We begin by stating the following:

**Definition IV.4.** Let \(\Omega(\tau) \subset \mathbb{C}\) be the closure of an open, bounded, simply connected set for \(\tau \geq 0\). Let \((u, \tau) \mapsto f(u, \tau), (u, \tau) \mapsto g(u, \tau)\) be the functions from \(\Omega(\tau) \times [\bar{\tau}, +\infty)\) to \(\mathbb{C}, 0 < \bar{\tau} < \infty\). Let \(f, g \in C^2(\Omega(\tau) \times I_1)\), where \(I_1\) is any compact subinterval of \([\bar{\tau}, +\infty)\), and let \(f, g\) be analytic in \(u \in \Omega(\tau)\). Then we say that \(f\) is uniformly approximated by \(g\) in \(\Omega(\tau)\) as \(\tau \to 0\) if there exist \(\alpha > 0, \gamma > 0, \tau_0 < \bar{\tau}\) independent of \((u, \tau)\) such that for all \(u \in \Omega(\tau)\) and \(\tau < \tau_0\),

\[
f(u, \tau) = g(u, \tau)(1 + \varepsilon(u, \tau)), \tag{4.14}
\]

where

\[
|\varepsilon(u, \tau)| + \left|\frac{de}{du}(u, \tau)\right| < \gamma \tau^\alpha.
\]

If \(\Omega_1, \ldots, \Omega_j\) are several such domains, then we say that \(f\) is uniformly approximated by \(g_1, \ldots, g_j\) on their union, provided (4.14) holds on each domain separately, and if \(C\) is a contour in such a domain or set of domains, we say that \(f\) is uniformly approximated on \(C\) by \(g_1, \ldots, g_j\).

**Remarks.** (1) It is easily seen that this is an equivalence relation: in particular we shall make use of the observation that if \(f\) is uniformly approximated by \(g\) and \(g\) is uniformly approximated by \(h\), then \(f\) is uniformly approximated by \(h\).

(2) Since Eq. (4.11) for \(\tau = 0, \theta = 0\) is the confluent hypergeometric equation in Whittaker's form (see, e.g., Buchholz [24]), the standard Picard approximation procedure yielding existence and uniqueness for the ODE Cauchy problem shows that with a suitable choice of normalization \(\phi'_1(\cdot, \tau, u)\) is uniformly approximated for \(u \in [0, 1]\) by the Whittaker function \(W_{\beta(m, k), m/2}(u)\). We remark that
$W_{\beta(m,k),m/2}(u)$ is an equivalent way of writing the unperturbed eigenvectors of Remark (3) after Proposition II.2, denoted by $\phi(m,k,u)$ in Proposition III.6:

$\phi(m,k,u) = W_{\beta(m,k),m/2}(u)$.

(3) Let $\Omega_1(\tau) = \{ u \in \mathbb{C} : \text{Re} \ u \geq r^{1/2}, \ \text{Im} \ u \geq 0, \ |u-2r| \geq r^{1/2} \}$. Then $\phi_1(\cdot, \tau, u)$ is uniformly approximated in $\Omega_1(\tau)$ by the JWKB-type function

$$\psi_-(\cdot, \tau, u) = K(\cdot, \tau) q'(\cdot, \tau, u) \exp \left( - \int_{t_1}^{u} q'(\cdot, u')^{1/2} du' \right), \quad (4.15)$$

where $t_1(m,k; \tau)$ is the zero of $q'(\cdot, \tau, u)$ near $\frac{1}{2} \left[ \beta_1^2(\cdot) + (\beta_1^2(\cdot))^2 + (m^2 - 1)/4 \right]^{1/2}$, and

$$K(\cdot, \tau) = r^{\beta_1(\cdot)} \sqrt{2} e^{-\sqrt{r}} \exp \left( \int_{t_1}^{\tau} q'(\cdot, u')^{1/2} du' \right). \quad (4.16)$$

The branch of the square root here and elsewhere is taken such that $\text{Re}(q'(\cdot))^{1/2} > 0$ as $u \to \infty$. Formulae (4.15) and (4.16) are immediate consequences of a theorem of Olver [33] and the estimate of the error control function given in Appendix B.

(4) When there are several domains of uniform approximation they may either touch at isolated points or overlap, and the overall approximating function may have jump discontinuities.

The foregoing remarks show that a uniform approximation has to be constructed only for $1 \leq u \leq \sqrt{r}$ and $0 < a \leq |u-2r| \leq \sqrt{r}$. To this end we apply the variation-of-parameters technique of Harrell and Simon [6]. The result is as follows:

**Lemma IV.5.** Let $\Omega_2(\tau) = C \cap \{ u : \text{Re} \ u \geq 2r - \sqrt{r} \}$, where $C$ is as in Proposition IV.3, and $\Omega_3(\tau) = \{ u : 1 \leq u \leq 2r \}$. Then:

1. For $u \in \Omega_3(\tau)$, $\phi_1^\alpha(\cdot, \tau, u)$ is uniformly approximated by $W_{\beta(m,k),m/2}(u)$ with $\alpha = 1/2$.
2. For $u \in \Omega_2(\tau)$, $\phi_1^\alpha(\cdot, \tau, u)$ is uniformly approximated by $\phi_-(\cdot, \tau, u)$ with $\alpha = 1/2$, where

$$\phi_-(\cdot, \tau, u) = T(\cdot, \tau) W_{-\beta(\cdot),m/2}(u-2r) + b(\cdot, \tau) W_{\beta(\cdot),m/2}(e^{i\alpha}(u-2r)), \quad (4.17)$$

$$T(\cdot, \tau) = 2K(\cdot, \tau)^2 \exp \left( - \int_{t_1}^{t_2} q'(m, \beta_1^2(\cdot), \tau, u) \ du \right) \times (1 + O(\cdot, \tau^{1/2})) \quad \text{as} \ \tau \to 0; \quad (4.18)$$

with $K(\cdot, \tau)$ as in (4.16), and

$$T(\cdot, \tau)^{-1} b(\cdot, \tau) = O(r^{\beta(\cdot, \tau)} e^{-\sqrt{r}}) \quad \text{as} \ \tau \to 0. \quad (4.19)$$
Proof. We first sketch the proof of (2). Following the variation-of-parameters technique of Harrell and Simon [6] (the reader is referred to that reference for a fully detailed description), for \( u \geq 2r + \sqrt{r} \), set

\[
\phi_-(\cdot, \tau, u) = K(\cdot, \tau, u) q'(m, \beta(m, k), \tau, u)^{-1/4} \cdot \exp \left( \int_{n_1}^u q'(m, \beta(m, k), \tau, u')^{1/2} du' \right),
\]

so that

\[
-\phi''_-(\cdot, u) + A(\cdot, u) \phi_-(\cdot, u) = 0, \quad u \geq 2r + \sqrt{r},
\]

for some function \((\tau, u) \mapsto A(\cdot, \tau, u)\) analytic in \(u\) and \(C^1\) in \(\tau\). Let \(\phi_-(\cdot, u)\) be \(C^1\) at \(u = 2r + \sqrt{r}\) and solve

\[
\left[ -\frac{d^2}{du^2} + \frac{1}{4} \frac{\beta(m, k)}{u - 2r} + \frac{m^2 - 1}{4(2r - u)^2} + \frac{m^2 - 1}{4} [(2r - u)^{-2} + 2u^{-1}(2r - u)^{-1}] \right] \phi_-(\cdot, u) = 0,
\]

where \(u\) belongs to \(C, 2r - \sqrt{r} \leq \text{Re} u \leq 2r + \sqrt{r}, \text{i.e., } \tilde{r} = \sqrt{r}\). Simple matching at \(u = 2r + \sqrt{r}\) with the use of the asymptotic formulae for Whittaker's functions (see, e.g., Abramowitz and Stegun [34], Buchholz [24]) shows that, on \(C \cap \{u: 2r - \sqrt{r} \leq \text{Re} u\},\)

\[
\phi_-(\cdot, \tau, u) = T(\cdot, \tau) W_{\beta(m, k), m/2}(u - 2r) + bW_{\beta(m, k), m/2}(e^{i\theta}(u - 2r)),
\]

where \(T(\cdot, \tau)\) is given by (4.18) and \(b(\cdot, \tau)/T(\cdot, \tau)\) satisfies (4.19). Furthermore, let \((u, \tau) \to \phi_+(\cdot, u, \tau)\) be defined as the unique function which satisfies (4.21) and is a simple multiple of \(W_{\beta(m, k), m/2}(e^{i\theta}(u - 2r))\) on \(C\). It is straightforward to check that \(W(\phi_-, \phi_+)_1 = 1\), where \(W(\cdot)\) denotes the Wronskian of \((\phi_-, \phi_+),\) and that

\[
B(\cdot, \tau, u) \equiv q'(\cdot, \tau, u) - A(\cdot, \tau, u) = 0(\cdot, \tau), \quad u \in C,
\]

\[
= 0(\cdot, (u - 2r)^{-2}), \quad u \geq 2r + \sqrt{r}.
\]

Furthermore, with the aid of the estimates on Whittaker's functions (see Buchholz [24] or Abramowitz and Stegun [34]) it is also easy to check that

\[
\int_u^\infty B(\cdot, \tau, u') \phi_+(\cdot, \tau, u') \phi_-(\cdot, \tau, u') \, du' = O(\cdot, \tau^{1/2}),
\]

\[
\int_u^\infty B(\cdot, u', \tau) \phi_-(\cdot, u', \tau)^2 \, du' = O(\cdot, \tau^{1/2}),
\]

\[
\int_u^\infty B(\cdot, u', \tau) \phi_+(\cdot, u', \tau)^2 \int_v^\infty B(\cdot, v, \tau) \phi_+^2(\cdot, v, \tau) \, dv \, du' = O(\cdot, \tau^{1/2}).
\]
Therefore it follows, as in Harrell and Simon [6], that on $\Omega_2(\tau)$
\[
\phi_1'(\cdot, \tau, u) = a_- (\cdot, \tau, u) \phi_- (\cdot, \tau, u) + a_+ (\cdot, \tau, u) \phi_+ (\cdot, \tau, u),
\]
\[
\frac{d}{du} \phi_1'(\cdot, \tau, u) = a_- (\cdot, \tau, u) \frac{d\phi_-}{du} (\cdot, \tau, u) + a_+ (\cdot, \tau, u) \frac{d\phi_+}{du} (\cdot, \tau, u),
\]
(4.26)
where $a_- (\cdot, \tau, u) = 1 + O(\cdot, \tau^{1/2})$, $a_+ (\cdot, \tau, u) = O(\cdot, \tau^{1/2})$. The same technique also proves that $\phi_1'(\cdot)$ is uniformly approximated by $W_{0(m, k), m/2}(u)$ on $[0, \sqrt{r}]$. This time use as comparison functions $\psi_- (\cdot)$ from (4.15), uniquely extended in a $C^1$-fashion to a linear combination of $W_{0(m, k), m/2}(u)$, $W_{0(m, k), m/2}(e^{i\pi}u)$ on $[1, \sqrt{r}]$, and $\psi_+ (\cdot) = \text{const} W_{-0(m, k), m/2}(e^{i\pi}u)$ on $[1, \sqrt{r}]$, extended to be a linear combination of $\psi_- (\cdot)$ and (dominantly) of $q' (\cdot)^{-1/4} \exp [\int_0^\tau q'(\cdot, u')^{1/2} du']$. Then a straightforward verification of (4.25) and the asymptotic formulae of Whittaker's functions show that $\phi_1'(\cdot, \tau, u)$ is uniformly approximated by $\psi_- (\cdot, \tau, u)$, which in turn uniformly approximated by $W_{0(m, k), m/2}(u)$ on $[1, \sqrt{r}]$. Since we already know that $\phi_1'(\cdot, \tau, u)$ is uniformly approximated by $W_{0(m, k), m/2}(u)$ on $[0, 1]$, the lemma is proved.

The estimate of the imaginary part is now easy to obtain:

**Proposition IV.6.** Let $(m, k)$ be fixed. Then, as $\tau \downarrow 0$,
\[
\text{Im} \beta_1'(m, k; \tau) = -\pi \frac{T(m, k; \tau)^2}{[k!(|m| + k)!]^2} (1 + O(\tau^{1/2}))
\]
\[
= \frac{-\pi(2r)^{2|m| + 4k + 2}}{[k!(|m| + k)!]^2} e^{-\tau^2/2}(1 + O(\tau^{1/2})).
\]
(4.27)

**Remark.** In the notation of Section II, by (4.8) formula (4.27) yields the behavior of $\text{Im} \beta_1'(m, k; e^{-in}\tau)$ as $\tau \to 0$. Furthermore, by the approximate dispersion-relation argument recalled in the proof of Corollary IV.2, integrating this time over the boundary of the circle $A_\varepsilon = \{ \tau : |\tau| = \varepsilon, 0 < \varepsilon < M_1(m, k) \}$ cut along the negative real axis, (4.27) yields the asymptotics of the coefficients $L_N(m, k)$,
\[
L_N(m, k) = [k!(|m| + k)!]^{-2}(N + 4k + 2|m| + 1)! (1 + O(m, k; N^{-1/2})).
\]
(4.28)

By the estimate of Harrell [13], it also yields the formula analogous to that of Brézin and Zinn-Justin (formula (4.4)) for the separation constant $\beta_2$,
\[
-\text{Im} \beta_1'(m, k; \tau e^{-in}) = \pi \Delta \beta_2(m, k, \tau)^2 (1 + O(m, k; \tau^{1/2})),
\]
(4.29)
Proof. \( \text{Im} \beta_1^*(m, k; \tau) \) is given by (4.12). By definition of \( t_2(m, k) \) and Lemma IV.5(1) we have

\[
\int_0^{t_2(m, k)} |\phi_1^*(m, k; \tau, u)|^2(u^{-1} + (2r - u)^{-1}) \, du
\]

\[
= \left[ \int_0^\infty W_{\beta(m, k), m/2}^2(u) u^{-1} \, du \right] \cdot (1 + O(m, k; \tau^{1/2}))
\]

\[
= \left[ (k!)^2 \int_0^\infty e^{-u} u^m (L_k^m(u))^2 \, du \right] (1 + O(m, k; \tau^{1/2}))
\]

\[
= k!(k + |m|)! \cdot [1 + O(m, k; \tau^{1/2})], \quad (4.30)
\]

where the well-known formulae on integrals of Whittaker and Laguerre functions (see Buchholz [24, pp. 23, 115]) have been used. Furthermore, by Lemma IV.5(2)

\[
\phi_1^*(m, k; \tau, t_2) \left. \frac{d}{du} \phi_1^*(m, k; \tau) \right|_{u = t_2} - \phi_1^*(m, k; \tau, t_2) \left. \frac{d}{du} \phi_1^*(m, k, \tau u) \right|_{u = t_2}
\]

\[
= T(m, k; \tau) \int W_{-\beta(m, k), m/2} W_{-\beta(m, k), m/2}(e^{-2\pi i u}) \right\} (1 + O(\cdot, \tau^{1/2})). \quad (4.31)
\]

Now, as proved in Appendix B,

\[
T(m, k; \tau) = (2r)^{2|m| + 1} \cdot 2^k e^{-2/\tau}(1 + O(\cdot, \tau^{3/2})) \quad (4.32)
\]

and (see Buchholz [24, p. 27])

\[
W\{ W_{-\beta(m, k), m/2}(u), W_{-\beta(m, k), m/2}(e^{-2\pi i u}) \}
\]

\[
= \frac{2\pi i e^{-\pi i \beta(m, k)}}{[\Gamma\left(\frac{m+1}{2} + \beta(m, k)\right)] [\Gamma\left(\beta(m, k) - \frac{m}{2}\right)]}
\]

\[
W\{ W_{-\beta(m, k), m/2}(u), W_{\beta(m, k), m/2}(e^{in} u) = -2\pi i [k!(|m| + k)!]. \quad (4.33)
\]

Inserting (4.30)–(4.33) into (4.12), we get (4.27). 

COROLLARY IV.7.

\[
\text{Im} \beta_1(m, j, \beta_1^*(m, k; \tau e^{-in}), \tau) \equiv \text{Im} \beta_1(m, j; \beta_1^*(m, k; \tau), \tau)
\]

\[
= \text{Im} \beta_1(m, j, k; \tau) = -2\tau \text{Im} \beta_1^*(m, k; \tau)(1 + O(\cdot, \tau)) \quad \text{as} \quad \tau \downarrow 0. \quad (4.34)
\]

Proof. Denoting the eigenvector \( \phi_1(m, j; \beta_1^*(\cdot, \tau), \tau) \) corresponding to
\( \beta_1(m, j, k; \tau) \) simply as \( \phi_1(\cdot) \), taking the imaginary part of the ODE \( t_m(\beta_1(\cdot, \tau), \beta_1'(\cdot, \tau), \tau) \phi_1(\cdot) = 0 \), multiplying by \( \phi_1(\cdot) \), and integrating, we get

\[
\text{Im} \beta_1(m, j, k; \tau) = -2 \int_0^{\infty} \frac{\text{Im} \beta_1'(m, k, \tau) \int_0^{\infty} |\phi_1(\cdot)|^2(2u + 2r)^{-1} du}{\int_0^{\infty} |\phi_1(\cdot)|^2u^{-1} du + \int_0^{\infty} |\phi_1(\cdot)|^2(2u + 2r)^{-1} du},
\]

whence (4.34) easily follows in the limit \( \tau \to 0 \). 

Proposition IV.8. As \( \tau \downarrow 0 \),

\[
\text{Im} \gamma_1(m, j, k; \tau) = \text{Im} \beta_1'(m, j, k; \tau)(1 + O(\cdot, \tau)),
\]

while for \( \tau \uparrow 0 \),

\[
\text{Im} \gamma_1(m, j, k; \tau) = \pi(-1)^m \frac{(j + 2k + |m| + 1)! (j + 2k + 2|m| + 1)!}{j!(k + |m|)!} \cdot 16(j + k + |m| + 1)4(2r)^{-2|m| - 2 - 4ke^{-2/|\tau|}}(1 + O(\cdot, |\tau|^{1/2})).
\]

Proof. For \( \tau \downarrow 0 \), i.e., \( \tau > 0 \), (4.35) is an immediate consequence of (4.32) by the definition of \( \gamma_1 \) (see Theorem III.2). For \( \tau < 0 \), i.e., \( \tau = |\tau| e^{\pm i \pi} \), once more by Theorem III.2 we can write

\[
\gamma_1(\cdot; \tau)|_{\tau < 0} = \beta_1(\cdot; \beta_1'(\tau e^{-i\pi}), \tau)|_{\tau < 0} + \beta_1'(\cdot; \tau e^{-i\pi})|_{\tau < 0}.
\]

Now \( \beta_1'(\cdot; \tau e^{-i\pi})|_{\tau < 0} = \beta_1'(\cdot; |\tau|) \) is real, and therefore \( \text{Im} \gamma_1(\cdot; \tau)|_{\tau < 0} = \text{Im} \beta_1(\cdot; \beta_1'(|\tau|), \tau)|_{\tau < 0} \), where the right side is defined in Corollary III.10. The argument leading to (4.36) is, up to the obvious modifications, identical to that of IV.5 and Proposition IV.6 applied this time to the limit as \( \text{Im} \theta \downarrow 0 \) of the equation (see (3.18))

\[
t_m(\beta_1, \beta_1'(\cdot; |\tau|), \theta, \tau) \phi_1(\cdot) = 0,
\]

and can therefore be omitted. 

Proof of Theorem IV.1. By (4.35), (4.36), and (4.27), as \( |\tau| \downarrow 0 \), \( \tau \in \mathbb{R} \),

\[
\text{Im} \gamma_1(m, j, k; \tau) = -\pi \frac{(2r)^{2|m| + 2 + 4ke^{-2/|\tau|}}}{[|k|(|m| + k)!]^2} (1 + O(\cdot, |\tau|^{1/2})).
\]

Now the inverse function \( \rho \to \Gamma_1(m, j, k; \rho) \) of \( \tau \to \tau \gamma_1(m, j, k, \tau)^{-1} \) exists and enjoys the properties stated in Theorem III.2(5). To see (4.1), it is enough to observe that with \( n = |m| + j + k + 1 \), by Propositions III.6(3) and III.10(2), we can write

\[
\tau \gamma_1(m, j, k; \tau)^{-1} = \tau n^{-1} + \tau^2 + O(\cdot, \tau^3) \quad \text{as} \quad |\tau| \downarrow 0,
\]
and thus $\Gamma_i(\cdot, \rho) = n\rho - n^3 \rho^2 + O(\cdot, \tau^3)$ as $|\tau| \downarrow 0$. Furthermore,

$$\begin{align*}
\text{Im}[ -\frac{i}{2} \gamma_i(\cdot, \tau)]^{-2} &= \frac{[\text{Re} \gamma_i(\cdot, \tau) \text{Im} \gamma_i(\cdot, \tau)]}{[(\text{Re} \gamma_i(\cdot, \tau))^2 + (\text{Im} \gamma_i(\cdot, \tau))^2]^2} \\
&= n^{-3} \text{Im} \gamma_i(\cdot, \tau)(1 + O(\cdot, \tau))
\end{align*}$$

by (4.37) and (4.36). Therefore (3.14) and (4.37) immediately yield (4.1). 

APPENDIX A

For the sake of completeness, in this appendix we prove some results about Borel summability of composed and implicit functions, because we do not know of any study where they may have been worked out before. We first prove that under certain circumstances Borel summability is stable under composition of functions.

**Proposition A.1.** Let $D = \{ z \in \mathbb{C} : 0 < |z| < M, \ |\text{arg } z| < \pi/2 \}$; let $x \mapsto f(x)$, $y \mapsto F(y)$ be analytic in $D$, continuous in $\bar{D}$, and let $f, F$ admit strongly asymptotic expansions as $x \to 0$, $y \to 0$, in $\bar{D}$, respectively, of the form

$$f(x) \sim x \sum_{n=0}^{\infty} a_n x^n,$$

$$|R_N(x)| \equiv \left| \frac{f(x)}{x} - \sum_{k=0}^{N-1} a_k x^k \right| \leq A^{N+1} N! |x|^N, \quad N = 1, \ldots,$$

$x \to 0$ in $\bar{D}$, $A$ independent of $x \in \bar{D}$,

$$F(y) \sim \sum_{i=0}^{\infty} b_i y^i,$$

$$|O_N(y)| \equiv \left| F(y) - \sum_{i=0}^{N-1} b_i y^i \right| \leq A_i^{N+1} N! |y|^N, \quad N = 1, \ldots,$$

$|y| \to 0$ in $\bar{D}$, $A_1$ independent of $y \in \bar{D}$.

Then $F \circ f = F(f(x))$ admits a strongly asymptotic expansion as $x \to 0$ in $\bar{D}$:

$$F(f(x)) \sim \sum_{l=0}^{\infty} c_l x^l,$$

$$|P_N(x)| \equiv \left| F(f(x)) - \sum_{l=0}^{N-1} c_l x^l \right| \leq C^{N+1} N! |x|^N, \quad N = 1, \ldots,$$

as $|x| \to 0$ in $\bar{D}$, with $C$ independent of $x \in \bar{D}$.

**Remarks.** (1) Our definition of strongly asymptotic expansion is that of Reed
and Simon [15, Sect. XII. 4]. We recall that by the Watson–Nevanlinna theorem (for further details see Sokal [26]) the stated analyticity bounds of the type (A.1), (A.2), (A.3) imply Borel summability for \(0 \leq x \leq A^{-1}, \ 0 \leq y \leq A^{-1}, \ 0 \leq x \leq C^{-1}\), respectively.

(2) The functions \(\rho \mapsto \Gamma_i(m, j; k; \rho)\) and \(\tau \mapsto \gamma_i(m, j; k; \tau)\) of Section II fulfill the conditions of \(f\) and \(F\), respectively.

Proof. In the sense of formal power series,

\[
\left( \sum_{k=0}^{\infty} a_k x^k \right)^2 = \sum_{n=0}^{\infty} a_n(2) x^n, \quad a_n(2) = \sum_{i=0}^{n} a_i a_{n-i},
\]

so

\[
|a_n(2)| \leq |2a_n a_0| + \sum_{i=1}^{n-1} |a_i a_{n-i}| \leq 2A^{n+2}n! + \sum_{i=1}^{n-1} \frac{i!(n-i)!}{n!} A^{n+2}
\]

\[
\leq 3A \cdot A^{n+1}n!
\]

by (A.1), since \(i!(n-i)!/n! \leq 1/n\). Iterating, we get

\[
\left( \sum_{n=0}^{\infty} a_n x^n \right)^i = \sum_{n=0}^{\infty} a_n(i) x^n, \quad i = 2,\ldots
\]

\[
|a_n(i)| \leq 3A^{(i-1)}A^{n+1}n!.
\]

Therefore \(F(f(x))\) has the asymptotic expansion

\[
F(f(x)) \sim \sum_{i=1}^{\infty} b_i x^i \left( \sum_{n=0}^{\infty} a_n x^n \right)^i \sim \sum_{i=1}^{\infty} b_i x^i \sum_{k=0}^{\infty} a_n^{(i)} x^k \sim \sum_{n=0}^{\infty} c_n x^n,
\]

\[
c_n = \sum_{i=0}^{n} a_n^{(i)} b_i.
\]

Now,

\[
|c_n| \leq \sum_{i=0}^{n} |b_i a_n^{(i)}| \leq A^1 A^{n+1} n! + \sum_{i=1}^{n} A^i + 1(3A)^{i-1} A^{n+1-i}(n-i)! n!
\]

by (A.4), and hence

\[
|c_n| \leq A^{n+2}n! + A^{n+2}(3A)^{n-1}2(n!) \leq (3A)^n A^{n+1} n!.
\]

Therefore (A.3) is implied by (A.2) if we insert (A.4) and (A.7) in (A.2) itself. 

Corollary A.2. Let \(x \mapsto f(x)\) be as in Proposition A.1, with strong asymptotic expansion \(\sum_{k=0}^{\infty} a_k x^k\), and let \((z, y, x) \mapsto F(z, y, x)\) be analytic in \((z, y, x) \in \{z: |z| \leq 1\} \times \{y: |y| < 1\} \times D, \) continuous in \(\overline{D}\) uniformly in \((z, y), \) and let \(F(z, y, x)\)
admit a strongly asymptotic expansion as \( x \to 0 \) in \( \bar{D} \) uniformly with respect to \( (z, y) \). Then the function \((z, x) \mapsto F(z, f(x), x)\) is analytic in \( \{z: |z| < 1\} \times \bar{D} \), continuous in \( \bar{D} \) uniformly with respect to \( z \), and admits a strongly asymptotic expansions as \( x \to 0 \) in \( \bar{D} \) uniformly with respect to \( z \).

Remark. The functions \((\beta_1, \beta_2, \tau) \mapsto \lambda(\cdot, \beta_1, \beta_2, \tau)\) and \(\tau \mapsto \beta_1'(m, k, e^{-i\tau}) - \beta(m, k)\) fulfill the conditions of \( F \) and \( f \), respectively.

**Proposition A.3.** Let \((y, x) \mapsto F(y, x)\) be as in Proposition A.2, and let \(x \mapsto \delta(x) = xf(x)\), where \(f(x)\) is analytic in \( D \), continuous in \( \bar{D} \), and admits the asymptotic expansion

\[
f(x) \sim \sum_{n=0}^{\infty} c_n x^n \quad \text{as} \quad x \to 0 \text{ in } \bar{D}.
\]

(A.8)

Then, if \( F(\delta(x), x) = 0, x \in \bar{D} \), the expansion (A.8) represents a strongly asymptotic expansion for \( x \mapsto f(x) \) in \( \bar{D} \).

Remarks. (1) The Borel summability statement for the inverse function is a particular case of this statement: it is enough to take \((y, x) \mapsto F(y, x) = F(y) - x\).

(2) The functions \((\beta_1', \tau') \mapsto \lambda(m, k; \beta_1', \tau')\) and \(\tau' \mapsto \beta_1'(m, k; \tau')\) satisfy the conditions of \( F \) and \( f \), respectively, so that \(\tau' \mapsto \delta(m, k; \tau') = \beta_1'(m, k; \tau') - \beta(m, k)\) satisfies the conditions of \( x \mapsto \delta(x) \). In fact, it suffices to rewrite the operator \(T'_n(\beta_1', \tau')\) as the action on \(D(T'_n(\cdot))\) of the differential expression

\[
\tilde{T}_n = \frac{d^2}{du^2} + \frac{\delta}{u} + \frac{\delta}{u + 2r'} - \frac{\beta(m, k)}{u} + \frac{\beta(m, k)}{u + 2r'}
\]

\[
+ \frac{m^2 - 1}{4} ((2r' + u)^2 - 2u^{-1}(2r' + u)^{-1})
\]

and to note that all its eigenvalues \(\lambda(m, k, \delta, \tau')\) are such that \((\text{cf. Proposition III.6}) (\partial \lambda / \partial \delta)(m, k; \delta, \tau')|_{\delta = 0, \tau' = 0} \neq 0\).

Proof. By assumption, \( F(\delta, x) \) admits the strongly asymptotic expansion \( F(\delta, x) = \sum_{i, k=0}^{\infty} a_{ik} x^{i} \delta^{k} \), with

\[
|a_{ik}| \leq B^{k} A^{i+1} i!
\]

(A.9)

for some \( B > 0, A > 0 \). Write

\[
f(x)^n = \sum_{k=0}^{\infty} c_k^{(n)} x^k, \quad c_0^{(0)} = \delta_{0,k}, \quad c_k^{(1)} = c_k, \quad k = 0, 1, \ldots
\]

(A.10)

\[
F(\delta(x), x) \sim \sum_{i, k=0}^{\infty} a_{ik} x^{i+k} \sum_{j=0}^{\infty} c_j^{(k)} x^j \equiv \sum_{n=0}^{\infty} d_n x^n,
\]

where

\[
d_n = \sum_{i=0}^{n} \sum_{k=0}^{n-i} a_{ik} c_k^{(n-k-i)}.
\]

(A.11)
We now prove that (A.9) and the equation \( F(\delta(x), x) = 0 \) imply the existence of constants \( D > 0, C > 0 \) such that

\[
|c_n| \leq DC^n n!. \tag{A.12}
\]

Let us proceed by induction. We have \( |c_0| < D \) for some \( D > 0 \). Assuming (A.12) true for \( k \leq n-2 \), let us prove it for \( k = n-1 \). Notice that if (A.12) is true up to \( k = n-2 \), then

\[
|c_{n-2}^{(k)}| \leq (3D)^{-1} DC^{n-2} (n-2)!. \tag{A.13}
\]

We now compute

\[
c_{n-1} = -(a_{01})^{-1} \left( \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} a_{ik} c_{n-k-i}^{(k)} + \sum_{i=0}^{n} a_{i0} c_{n-i}^{(0)} + \sum_{k=0}^{n} a_{0k} c_{n-k}^{(k)} \right)
\]

\[
= -(a_{01})^{-1} \left( \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} a_{ik} c_{n-k-i}^{(k)} + a_{n0} + \sum_{k=2}^{n} a_{0k} c_{n-k}^{(k)} \right).
\]

Hence

\[
|c_{n-1}| \leq |a_{01}|^{-1} \left( \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} (3D)^{-1} DC^{n-k-i} (n-k-i)!ight)
\]

\[
+ A^{n-1} n! + \sum_{k=2}^{n} AB^{k} (3D)^{-1} DC^{n-k} (n-k)!ight)
\]

\[
\leq AB|a_{01}|^{-1} \left( \sum_{i=1}^{n-1} A_i i! \sum_{n=1}^{n-i} (3DB)^{-1} C^{k-i} C^{-k+i+1} \frac{(n-k-i)!}{(n-1)!} \right)
\]

\[
+ \left( \frac{A}{C} \right)^{-1} \left( A/D \right) n B \sum_{k=2}^{n} (3BD)^{-1} C^{-k-1} C^{-(k-1)} \frac{(n-k)!}{(n-1)!} DC^{n-1} (n-1)!ight)
\]

\[
\leq AB|a_{01}|^{-1} DC^{n-1} (n-1)! \left( \sum_{i=1}^{n} \left( \frac{A}{C} \right)^i i! \sum_{j=0}^{n-i-1} \left( \frac{3BD}{C} \right)^j \frac{1}{j!} \frac{j! (n-i-j)! (n-i-1)!}{(n-i-1)! (n-1)!} \right)
\]

\[
+ \sum_{j=0}^{n-2} \left( \frac{3BD}{C} \right)^j \frac{1}{j!} \frac{(n-j)! (n-2-j)!}{(n-2)! (n-1)!} \right)
\]

\[
\leq AB|a_{01}|^{-1} DC^{n-1} (n-1)! \left( \sum_{i=1}^{n} \left( \frac{A}{C} \right)^i i! \frac{(n-i-1)!}{(n-1)!} \right)
\]

\[
+ \left( \frac{A}{D} \right) \frac{A}{C} n^{-1} B + \frac{3}{(n-1)} \left( \frac{3BD}{C} \right) e^{(3BD/C)}
\]
\[ \leq AB|a_{01}|^{-1}DC^{n-1}(n-1)! \left( \sum_{j=0}^{n-1} \left( \frac{A}{C} \right)^{j+1} \frac{(j+1)!(n-1-j)!}{(n-1)!} \right) \cdot (3e)^{(3BD/C)} + \left( \frac{A}{D} \right) \left( \frac{n}{b} + \frac{3}{(n-1)!} \right) \left( \frac{3BD}{C} \right) e^{(3BD/C)} \]

\[ \leq AB|a_{01}|^{-1}(n-1)! \left( 9 \left( \frac{A}{C} \right)^{n} \right) (3e)^{(3BD/C)} + \left( \frac{A}{D} \right) \frac{n}{B} \]

\[ + \frac{9BD}{(n-1)!} \left( \frac{3BD}{C} \right) e^{(3BD/C)} \leq DC^{n-1}(n-1)! , \]

if we choose \( 1 < A, B \leq D \leq C \), since by assumption

\[ \left| F(\delta, x) - \sum_{i,k=0}^{N-1} a_{ik} x^i \delta^k \right| \leq B^N A^{N+1} |\delta|^N |x|^N N! \]

as \( x \to 0, x \in \bar{D} \), (A.11) and (A.12) imply that

\[ \left| f(x) - \sum_{k=0}^{n-1} c_n x^n \right| \leq DC^N N! |x|^n \quad \text{as} \quad x \to 0 \text{ in } \bar{D} , \]

which proves the assertion. \[ \]

**APPENDIX B**

In this appendix we compute the tunneling factor \( T(\cdot) \) used in (4.17), (4.18) and bound the error-control function needed to justify formulae (4.15) and (4.16).

We begin with the error-control function, which is the total variation of

\[ q'(\cdot, u)^{-1/4} \frac{d^2}{du^2} q'(\cdot, u)^{-1/4} = \frac{1}{4} \left( \frac{d^2}{du^2} q'(\cdot, u) \right) q'(\cdot, u)^{-3/2} \]

\[ + \frac{5}{16} \left( \frac{d}{du} q'(\cdot, u) \right)^2 q'(\cdot, u)^{-5/2} \] (B.1)

for \( r^{1/2} \leq u \leq 2r - r^{1/2} \). It has to be shown that this quantity tends to 0 as \( r \to \infty \), i.e., \( \tau \to 0 \). Now, from the definition of \( q'(\cdot, u) \) in (4.7) with \( \theta = 0 \), it is easy to see that, uniformly in \( u \), \( r^{1/2} < u < 2r - r^{1/2} \), \( q'(\cdot, u)^{-1} = O(1) \), \( (d/du) q'(\cdot, u) = O(\tau) \), \( (d^2/du^2) q'(\cdot, u) = O(\tau^{3/2}) \) as \( \tau \downarrow 0 \). Thus

\[ q'(\cdot, u)^{-1/4} \frac{d^2}{du^2} q'(\cdot, u)^{-1/4} = O(\tau^{3/2}) \quad \text{as} \quad \tau \downarrow 0. \]

Since \( q'(\cdot, u) \) is a rational function of \( u \) and \( \tau \), the total variation of this quantity is also the integral of a function \( O(\tau^{3/2}) \), and is thus \( O(\tau^{1/2}) \).

Next we estimate \( K(\cdot) \) and \( T(\cdot) \), defined in (4.16) and (4.18). We claim:
Proposition B.1.

\[ T(m, k; \tau) = 2\tau^{-(|m|+2k+1)}e^{-1/\tau}(1 + O(\cdot, \tau^{1/2})) \quad \text{as} \quad \tau \downarrow 0. \]

Proof. Because of the uniformity of the approximations, it suffices to determine \( T \) by asymptotic matching. The quantity \( K(\cdot) \) of (4.16) is determined to leading order by the condition that

\[ K(m, k; \tau) q'(\cdot, \beta(\cdot, \tau), \tau, u)^{-1/4} \exp\left(-\int_{t_1}^{u} q'(\cdot, u') \, du'\right) \]

\[ = W_{\beta(m,k),m/2}(u) \cdot (1 + O(\cdot, \tau^{1/2})) \]

at \( u = \sqrt{r} \) (say). Thus we may set

\[ K(m, k; \tau) = \tau^{-\beta(m,k)/2} e^{-1/2\tau^{1/2}} \exp\left(\int_{t_1}^{\tau^{1/2}} q'(\cdot, \tau, u') \, du'\right), \tag{B.2} \]

with the aid of an expansion of Buchholz [24]. Then \( T(\cdot) \) is determined by

\[ T(m, k; \tau) = 2[K(m, k; \tau)]^2 \exp\left(-\int_{t_1}^{\tau} [q'(\cdot, \tau, u')]^{1/2} \, du'\right) \cdot (1 + O(\cdot, \tau^{1/2})). \]

Since

\[ \int_{t_1}^{\tau^{1/2}} q'(\cdot, \tau, u')^{1/2} = \int_{2r-\sqrt{r}}^{2r} q'(\cdot, \tau, u')^{1/2} \, du', \]

we get

\[ T(m, k; \tau) \tau^{-\beta(m,k)} e^{-\tau^{-1/2}} \exp\left(-\int_{\sqrt{r}}^{2r-\sqrt{r}} q'(\cdot, u')^{1/2} \, du'\right) (1 + O(\cdot, \tau^{1/2})) \]

\[ = \tau^{-\beta(m,k)} e^{-\tau^{-1/2}} \exp\left(-2\int_{\sqrt{r}}^{r} \left(\frac{1}{4} - \beta(\cdot) u^{-1} - \beta(\cdot)(2r-u)^{-1}ight.ight. \]

\[ \left. + \frac{m^2-1}{4} (u^{-1} + (2r-u)^{-2}) \, du\right) (1 + O(\cdot, \tau^{1/2})) \]

\[ = \tau^{-\beta(\cdot)} e^{-\tau^{-1/2}} \exp\left(-\int_{\sqrt{r}}^{r} (1 - 2\beta(\cdot) u^{-1} - 2\beta(\cdot)(2r-u)^{-1}) \, du\right) \]

\[ \cdot (1 + O(\cdot, \tau^{1/2})) \]

\[ = \tau^{-\beta(\cdot)} \exp(-\tau^{-1/2}) \exp(\tau^{-1} + \tau^{-1/2} + 2\beta(\cdot) \ln(\tau^{1/2}) \]

\[ + 2\beta(\cdot) \ln(2\tau^{-1/2} - 1)) \cdot (1 + O(\cdot, \tau^{1/2})) \]

\[ = \left(\frac{\tau}{2}\right)^{-2\beta(m,k)} e^{-\tau^{-1}} (1 + O(\cdot, \tau^{1/2})). \]

REFERENCES


LIST OF SYMBOLS

\begin{align*}
\alpha & \quad \text{Eq. (4.26)} \\
A & \quad \text{Eq. (4.21)} \\
A_n & \quad \text{below Eq. (3.13)} \\
A' & \quad \text{below Eq. (3.13)} \\
B & \quad \text{Eq. (4.24)} \\
B_n & \quad \text{below Eq. (3.13)} \\
B'_n & \quad \text{below Eq. (3.13)} \\
C & \quad \text{Lemma IV.5}
\end{align*}
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