Connection formula, hyperasymptotics, and Schrödinger eigenvalues: dispersive hyperasymptotics and the anharmonic oscillator

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1 Introduction

In recent years there has been a resurgence of interest in the high-order terms of asymptotic expansions of physical quantities [1]–[5]. Traditionally summation of asymptotic power series involves truncation before the smallest term. The residual error is connected to the closest singularity of the Borel transform on the same Riemann sheet [6]. There are often additional, exponentially small terms, neglected by the Poincaré definition of asymptotic power series, but whose contributions may be numerically comparable and analytically essential to the reconstruction of the function that gave rise to the series. Hyperasymptotics refers to a sequence of approximations that transcend the truncation error of the main series while using the associated exponentially small series [1].

The possibility of exponential accuracy was noted by Stieltjes [7] and explored by several authors [8, 9]. The idea of systematic exponential approximations was first conceived by Dingle [10] (later rediscovered by others, e.g., Raković and Solov’ev [11]), continued by field theorists [12], and culminated in a major mathematical treatment by Écalle [13]. A practical result is a procedure that “decodes” the late terms, allowing a (partial) reconstruction of the function in calculable form, an iterative, calculational procedure called hyperasymptotics [1, 2],[14]–[19].

In this paper we apply hyperasymptotics systematically to a single eigenvalue problem, the quarticly perturbed one-dimensional harmonic oscillator. For each energy level there is an asymptotic power series—the usual Rayleigh-Schrödinger perturbation series—and a sequence of exponentially smaller subseries that can be derived formally by semiclassical methods. (Recently, Kowalenko and Rawlinson [20] applied the Mellin-Barnes regularization of Ref. [18] to the Rayleigh-Schrödinger power series, obtaining the first and second terms of what we call the first-level hyperasymptotic expansion, but apparently unaware of higher-order semiclassical results on the anharmonic oscillator, they could not proceed further.) The computational connection between one subseries and the next, which we justify heuristically, has the form of a dispersion relation that arises by contour deformation of a Cauchy integral in the coupling constant plane. (One might expect involvement of the Bender-Wu branch cuts [21, 22], but such a link is not made in this paper.)

Although our emphasis is the anharmonic oscillator, the technique of hyperasymptotic approx-
imations for functions that satisfy a dispersion relation is general. (See, e.g., Refs. [14, 23, 24].) In this sense, our work is analogous to Boyd’s hyperasymptotic expansions for Bessel functions via the Stieltjes transform [23].

A more detailed exposition will be published elsewhere [25].

2 Airy function paradigm

Key to the invention of hyperasymptotics was an integral for the remainder after truncation of the asymptotic power series. In this section we show how hyperasymptotics develop from dispersion relations with the Airy $\text{Ai}(z)$ previously treated by Berry and Howls [1], but without using the explicit integral representation for $\text{Ai}(z)$. The only information about $\text{Ai}(z)$ used is its basic asymptotic expansion [Eqs. (1) and (5)] and dispersion relation [Eq. (6)]. The asymptotics of $\text{Ai}(z)$ are uncomplicated (in contrast to the oscillator) and permit us to focus on general technique.

2.1 Asymptotic power series for the Airy function

The fundamental asymptotic expansion for $\text{Ai}(z)$ is a multiplicative factor times a divergent series in negative powers of $z^{3/2}$ [Eq. (10.4.59) of Ref. [26]; for a detailed discussion of the domains of validity, see Ref. [27]]:

$$\text{Ai}(z) \sim \frac{1}{2} \pi^{-1/2} \left( \frac{3}{4} \xi \right)^{-1/6} e^{-\xi/2} \sum_{k=0}^{\infty} (-1)^k c_k \xi^{-k}, \quad (|\arg z| < 2\pi/3),$$  \hspace{1cm} (1)

$$\xi = \frac{3}{4} z^{3/2}, \quad c_k = \Gamma \left( \frac{1}{6} + k \right) \Gamma \left( \frac{5}{6} + k \right) / \Gamma \left( \frac{1}{6} \right) \Gamma \left( \frac{5}{6} \right) k!.$$  \hspace{1cm} (2)

We use a definition of $c_k$ that is $2^k$ times the $c_k$ of Abramowitz and Stegun [26], with a corresponding factor of 2 difference in the expansion variable $\xi$. All other related asymptotic expansions for $\text{Ai}(z)$ and its companion $\text{Bi}(z)$ are combinations of Eq. (1) and the continuation formulas,

$$2e^{i\pi/6} \text{Ai} \left( e^{\pm 2\pi i/3} z \right) = \text{Bi}(z) \pm i\text{Ai}(z).$$  \hspace{1cm} (3)

It is convenient to define a “dressed” Airy function $A(\xi)$, whose asymptotic expansion is just the power series factor in Eq. (1),

$$A(\xi) \equiv 2\pi^{1/2} \left( \frac{3}{4} \xi \right)^{1/6} e^{\xi/2} \text{Ai}(z),$$  \hspace{1cm} (4)

$$A(\xi) \sim \sum_{k=0}^{\infty} (-1)^k c_k \xi^{-k}, \quad (|\arg \xi| < \pi).$$  \hspace{1cm} (5)

2.2 Dispersion relation and asymptotic expansion

$A(\xi)$ satisfies a “once-subtracted” dispersion relation,

$$A(\xi) = \frac{\xi}{2\pi} \int_{0}^{\infty} d\tau \frac{e^{-\tau}}{\tau (\xi + \tau)} A(\tau),$$  \hspace{1cm} (6)
that is essentially a manipulation of the Cauchy formula,
\[
\frac{A(\xi) - 1}{\xi} = \frac{1}{2\pi i} \oint dt \frac{A(t) - 1}{t(t - \xi)},
\]
(7)
and of the discontinuity of \(A(t)\) across the negative \(i\) axis that is provided by the continuation formula (3),
\[
A(e^{i\pi} \tau) - A(e^{-i\pi} \tau) = i e^{-\tau} A(\tau).
\]
(8)

The geometric sum,
\[
\frac{\xi}{\tau(\xi + \tau)} = \frac{1}{\tau} \left[ 1 - \frac{\tau}{\xi} + \frac{\tau^2}{\xi^2} - \frac{\tau^3}{\xi^3} + \cdots \right. \\
\left. + (-1)^{N-1} \frac{\tau^{N-1}}{\xi^{N-1}} + (-1)^N \frac{\xi}{\xi^N} \right],
\]
leads to a family of subtracted dispersion equations—\(N\) can be any positive integer or zero:
\[
A(\xi) = \sum_{k=0}^{N-1} c_k (-1)^k \xi^{-k} + R(N, \xi),
\]
(10a)
\[
R(N, \xi) = \frac{(-1)^N \xi^{1-N}}{2\pi} \int_0^\infty d\tau \frac{\tau^{N-1} e^{-\tau}}{\xi + \tau} A(\tau).
\]
(10b)

The asymptotic expansion results from dropping the remainder term \(R(N, \xi)\). We shall see in the next subsection that Eq. (10) is the engine that generates the hyperasymptotic expansions for \(A(\xi)\).

### 2.3 Hyperasymptotic expansions for \(A(\xi)\)

The hyperasymptotic expansions for \(A(\xi)\) result from substituting the dispersion relation (10) recursively into itself. The value of \(N\) can be chosen differently at each step. For instance, substituted twice into itself, Eq. (10) yields a second-level hyperasymptotic expansion plus remainder:
\[
A(\xi) = \sum_{k_0=0}^{N_0-1} c_{k_0} (-1)^{k_0} \xi^{-k_0} + \frac{(-1)^{N_0} N_1-1}{2\pi \xi^{N_0}} \sum_{k_1=0}^{N_1-1} c_{k_1} (-1)^{k_1} T(N_0 - k_1, \xi) \\
+ \frac{(-1)^{N_0+N_1}}{2\pi^2 \xi^{N_0}} \sum_{k_2=0}^{N_2-1} c_{k_2} (-1)^{k_2} T(N_0 - N_1, N_1 - k_2, \xi) \\
+ R(N_0, N_1, N_2, \xi).
\]
(11)

The first-level \(T\) depend on \(N_0 - k_1\) and \(\xi\), but not on \(A(\xi)\) (and can be expressed in terms of the gamma and incomplete gamma functions),
\[
T(N_0 - k_1, \xi) \equiv \xi \int_0^\infty d\tau_1 \frac{\tau_1^{N_0-k_1-1} e^{-\tau_1}}{\xi + \tau_1}.
\]
(12)
The second-level \(T\) functions [also independent of \(A(\xi)\)] are defined by
\[
T(N_0 - N_1, N_1 - k_2, \xi) \equiv \xi \int_0^\infty d\tau_1 \tau_1^{N_0-N_1} \frac{e^{-\tau_1}}{\xi + \tau_1} \int_0^\infty d\tau_2 \tau_2^{N_1-k_2-1} \frac{e^{-\tau_2}}{\tau_1 + \tau_2}.
\]
(13)
The second-level remainder is a triple-integral. The next higher-level hyperasymptotic expansion follows by substituting Eq. (10) into the remainder term of Eq. (11), and so forth. In this manner, dispersion relations generate successively higher-order hyperasymptotic expansions.

The utility of hyperasymptotic expansions is that the remainders $R(N_0, \ldots, N_p, \xi)$ can often be made successively smaller by appropriate choice of $N_0, \ldots, N_p$. The lower-order $c_k$ in effect sum the omitted higher-order terms of the preceding-level hyperasymptotic series. Successively higher accuracy is extracted from the divergent power-series coefficients at the cost of calculating the higher-level $T$'s (for methods, see Refs. [1, 17, 24]).

The Airy function is simple in that it appears in its own dispersion relation. More typically the function on the right differs from the left, and Eq. (10) might be replaced by a sequence of equations. This is the case for the anharmonic oscillator eigenvalues, where several more complicated scenarios occur.

3 Anharmonic Oscillator

The $n$th energy eigenvalue $E_n(g)$ of the quarticly anharmonic oscillator, whose Schrödinger equation is

$$\left[ -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 + g x^4 - E_n(g) \right] \psi(x) = 0,$$  

(14)

has a factorially divergent but Borel-summable [28] asymptotic power series in $g$, given by Rayleigh-Schrödinger perturbation theory (RSPT):

$$E_n(g) \sim E_n^{RSPT}(g) \equiv \sum_{j=0}^{\infty} E_n^{(j)} g^j.$$  

(15)

The energy has a cut on the negative real $g$ axis, and the once-subtracted dispersion relation holds [21, 29],

$$E_n(g) = E_n^{(0)} + \frac{(-g)}{2\pi i} \int_0^{\infty} \frac{\Delta E_n(z)}{z(z + g)} dz,$$  

(16)

where $E_n^{(0)} = n + 1/2$, and where

$$\Delta E_n(z) \equiv E_n(e^{-i\pi} z) - E_n(e^{+i\pi} z)$$  

(17)

denotes the discontinuity of $E_n(g)$ across the negative $g$ axis when $g = -z$. Factorial divergence and dispersion relation suggest hyperasymptotics.

3.1 General expansion for the energy that includes all exponentially small orders

We sketch the derivation of the asymptotic expansion of the anharmonic oscillator energy through second-exponentially-small order following the strategy of Ref. [30].
First change variables:

\[ \sigma = x^2, \quad \psi(x) = \sigma^{-1/4} \Phi(\sigma), \]  

\[ \left[ -\sigma \frac{d^2}{d\sigma^2} - \frac{3}{16\sigma} + \frac{1}{4} \sigma + \frac{1}{2} g\sigma^2 - \frac{1}{2} E(g) \right] \Phi(\sigma) = 0. \]

Using comparison equation ideas of Langer [31] and Cherry [32], one builds two asymptotic expansions, one “anchored” at the origin, involving Whittaker’s confluent hypergeometric function [26], the other “anchored” at the outer turning point, involving Airy functions. These expansions have to be matched in the intermediate region, which yields an asymptotic expansion for the energy eigenvalue that looks like RSPT [Eq. (15)], but with an important difference: the RSPT \( E_n^{(j)} \) of Eq. (15) is a polynomial of degree \( j + 1 \) in the harmonic oscillator quantum number \( n \); here the \( n \) is replaced by a function \( w(g) \) whose value is taken from the first index of the Whittaker function, which in turn is \( \frac{1}{2} w + \frac{1}{4} \).

\[ E_n(g) \sim \sum_{j=0}^{\infty} E_n^{(j)} g^j = E_w^{\mathrm{RSPT}}(g), \]  

\[ E_w^{(0)} = w + \frac{1}{2}, \]  

\[ E_w^{(1)} = \frac{3}{2} w^2 + \frac{3}{2} w + \frac{3}{4}, \]  

\[ E_w^{(2)} = -\frac{17}{4} w^3 - \frac{51}{8} w^2 - \frac{89}{8} w - \frac{21}{8}, \]  

\[ E_w^{(3)} = \frac{375}{16} w^4 + \frac{375}{8} w^3 + \frac{177}{2} w^2 + \frac{1941}{16} w + \frac{389}{16}. \]

If \(|\arg g| < \pi\), the matching process implies that \( w = n \), and Eq. (20) is identical to the RSPT Eq. (15). If \( \mp \arg g = \pi + \epsilon > \pi \), the matching process implies that

\[ w = n + \Delta w_\mp(n, g'), \quad (g = e^{\mp i\pi} g', \quad \mp \arg g' = \epsilon > 0). \]

That is \(|\arg g| = \pi\) is a Stokes line for the energy eigenvalue: the RSPT expansion changes by replacement of \( n \) with \( n + \Delta w_\mp \), depending on the phase of \( g \).

The equation satisfied by \( \Delta w_\mp(n, g') \) is [cf. Ref. [30], Eq. (50)]:

\[ \Delta w_\mp(n, g') = \mp \frac{i}{\pi} \ln [1 + \pi B(n + \Delta w_\mp, g') f(n + \Delta w_\mp, g')], \]  

\[ = \mp \frac{i}{\pi} \ln \left\{ 1 + \pi \exp(\Delta w_\mp \partial / \partial n) [B(n, g') f(n, g')] \right\}, \]

\[ (g = e^{\mp i\pi} g', \quad \arg g' = \mp \epsilon, \quad \epsilon > 0), \]

where \( B(w, g') = 2\pi C_w(3g')^{-w - \frac{1}{2} e^{-\frac{1}{2} g'}} \),

\[ C_w = \frac{12w + \frac{1}{2}}{\pi \sqrt{2\pi^2 (w + 1)}}, \]

and where the function \( f \), generated by the matching, has a power series expansion in \( g' \) with polynomial coefficients in \( w \),

\[ f(w, g') = 1 - g' \left( \frac{69}{24} + \frac{17}{4} w + \frac{17}{4} w^2 \right) + g'^2 \left( -\frac{9011}{1152} - \frac{1829}{96} w - \frac{95}{24} w^2 + \frac{39}{16} w^3 + \frac{289}{32} w^4 \right) + O(g'^3). \]
In Eq. (27), $\Delta w_\mp$ is held constant with respect to the $\partial/\partial n$. We point out that the full matching condition Eq. (27) can be equivalently derived via analytic continuation of suitably normalized Borel-summable JWKB expansions. In this geometric setting, our index $w$ appears as twice a “rescaled monodromy exponent $s$” that describes the change of the JWKB wavefunction after traversing a loop around the double turning point at the origin, while the equivalent of our $f(w, g')$ is computed via an “exact matching method” that again reduces to a comparison with Weber’s differential equation [33, 34].

Equations (27) and Eq. (28) permit a recursive, iterative solution as a sum of exponentially small terms, $\Delta w^{(k)}_\mp \sim e^{-\frac{k}{3z}}$:

\[
\Delta w^{(1)}_\mp (n, z) = \Delta w^{(2)}_\mp + \Delta w^{(3)}_\mp + \cdots,
\]

\[
\Delta w^{(1)}_\mp = iB(n, z)f(n, z),
\]

\[
\Delta w^{(2)}_\mp = \pm \frac{i\pi}{2} B^2 f^2 - Bf \frac{\partial}{\partial n} Bf.
\]

The expansion for $\Delta E_n(z)$, induced by Eqs. (31)–(33), depends on arg $z$ and breaks into exponentially small $[e^{-k/3z}]$ sub-expansions. With $(g = e^{\mp i\pi z}, \mp \text{arg } z = \epsilon > 0)$, one obtains,

\[
\Delta E_n(z) = \Delta E^{(1)}_n(z) + \left[i \Delta E^{(2,i)}_n(z) \pm \Delta E^{(2,r,\mp)}_n(z)\right] + \cdots,
\]

\[
\Delta E^{(1)}_n(z) = -iB(n, z)f(n, z) \frac{\partial E^{\text{RSPT}}_n(-z)}{\partial n},
\]

\[
i \Delta E^{(2,i)}_n(z) = i\frac{\pi}{2} B(n, z)^2 f(n, z)^2 \frac{\partial E^{\text{RSPT}}_n(-z)}{\partial n},
\]

\[
\Delta E^{(2,r,\mp)}_n(z) = B^2 \left(f^2 \frac{\partial E^{\text{RSPT}}_n(-z)}{\partial n} \right) \left\{ f^2 \left[\ln \left(\frac{1}{3} z\right) + \psi(n + 1)\right] - f \frac{\partial f}{\partial n}\right\} - \frac{1}{2} B^2 f^2 \frac{\partial^2 E^{\text{RSPT}}_n(-z)}{\partial n^2}.
\]

The notation $\Delta E^{(2,r,\mp)}_n(z)$ reflects choosing the upper signs in: $\Delta E^{(2,r)}_n(z) = \pm \Delta E^{(2,r,\mp)}_n(z)$, depending whether $\mp \text{arg } z = \epsilon > 0$.

$f(n, g')$ is the unpleasantly complicated expression that follows the prefactor $-iB(n, g')$ in Eq. (50) of Ref. [30]. Remarkably (cf. Refs. [25, 35]) it has a simple, direct relation to the RSPT coefficients:

\[
\ln f(n, z) = \sum_{k=1}^{\infty} z^k \frac{1}{3k} \left(-1\right)^{k+1} \frac{d}{dn} E^{(k+1)}_n.
\]

Thus, $\Delta E^{(1)}_n(z)$ is $-iB(n, z)$ times a power series in $z$ with constant term 1 and real coefficients; $i\Delta E^{(2,i)}_n(z)$ is $(i\pi/2)B(n, z)^2$ times a real power series in $z$; $\Delta E^{(2,r,\mp)}_n$ involves two real power series; and all the coefficients $b^{(k)}_n$, $c^{(k)}_n$, and $d^{(k)}_n$ follow from the RSPT $E^{(k)}_n$:

\[
\Delta E^{(1)}_n(z) = -i \left[ 2\pi C_n(3z)^{-n} - \frac{1}{2} e^{-\frac{1}{3} z} \right] \sum_{k=0}^{\infty} b^{(k)}_n (3z)^k,
\]

\[
i \Delta E^{(2,i)}_n(z) = i\frac{\pi}{2} \left[ 2\pi C_n(3z)^{-n} - \frac{1}{2} e^{-\frac{1}{3} z} \right] \sum_{l=0}^{\infty} c^{(l)}_n (3z)^l.
\]
\[ \Delta E_n^{(2,r_{-1})}(z) = B(n, z)^2 \left\{ \left[ \ln\left(\frac{1}{3} z\right) + \psi(n + 1) \right] \sum_{l=0}^{\infty} c_n^{(l)} (3z)^l - \frac{1}{2} \sum_{l=1}^{\infty} d_n^{(l)} (3z)^l \right\}. \] (41)

Numerical values of \( b_n^{(k)} \) through 50th order were tabulated for \( n = 0, 1, 2 \) in Ref. [30]. The \( c_n^{(k)} \) and \( d_n^{(k)} \) are tabulated in Ref. [25].

### 3.2 Stokes line: \( \Delta E_n^{(2)} \); dispersion relation for \( b_n(z) \)

From the explicit discontinuity in Eq. (34), one infers that \( z > 0 \) is a Stokes line for \( \Delta E_n^{(2)}(z) \). Its use in Eq. (16) requires explanation.

We note that \( E_n(g) \) satisfies a Schwarz reflection principle: \( \Delta E_n(z) = E_n(e^{-i\pi} z) - E_n(e^{i\pi} z) \) is purely imaginary for \( z \) real and positive. In fact, \( \Delta E_n(z) \) is analytic on \( z > 0 \). In the expansion \( \Delta E_n^{(1)} \) [Eq. (39)], the late \( b_n^{(k)} \) grow (faster than) factorially with \( k \) and have the same sign (see Eqs. (71) and (72) of Ref. [30], and also Ref. [36]),

\[ b_n^{(k)} \sim -2C_n \Gamma \left( k + n + \frac{1}{2} \right) \ln \left( k + n + \frac{1}{2} \right). \] (42)

Consequently, the Borel sum of the power series, denote by it \( b_n(z) \),

\[ b_n(z) = \text{Borel Sum of } \sum_{k=0}^{\infty} b_n^{(k)} (3z)^k, \] (43)

has a cut on the positive \( z \) axis with purely imaginary, exponentially small discontinuity. After multiplication by the appropriate exponentially small prefactor, one has a real, second-exponentially-small discontinuity in the Borel sum of \( \Delta E_n^{(1)}(z) \) [Eq. (39)], seemingly in conflict with the Schwarz principle and the continuity of \( \Delta E_n(z) \). The explicit \( \pm \Delta E_n^{(2,r_{-1})}(z) \) in Eq. (34) must cancel (to that order) the discontinuity. (Such a cancellation was also noted by Zinn-Justin [37] as a relation between instanton contributions in the multi-instanton expansion for potentials with symmetric degenerate minima.)

Define \( \Delta b_n(z) \) when \( z \) is real and positive, and by analytic continuation when \( z \) is complex, by

\[ \Delta b_n(z) \equiv b_n(z - i0) - b_n(z + i0), \quad \text{for } z > 0. \] (44)

We conjecture that \( b_n(z) \) satisfies the subtracted dispersion relations,

\[ b_n(z) = \sum_{k=0}^{N_1-1} b_n^{(k)} (3z)^k - \frac{z^{N_1}}{2\pi i} \int_0^\infty dz' \frac{z'^{-N_1} \Delta b_n(z')}{z' - z}, \] (45)

and that \( \Delta E_n^{(2,r_{-1})}(z) \) provides an asymptotic expansion for \( \Delta b_n(z) \),

\[ \Delta b_n(z) \sim \frac{-2i}{2\pi C_n} (3z)^{n+\frac{1}{2}} e^{\frac{1}{3z} \Delta E_n^{(2,r_{-1})}(z)}. \] (46)

By Eqs. (45) and (46), the infinite summation in Eq. (39) becomes a partial sum plus dispersion-
The addition of \( \pm \Delta E_n^{(2,r,-)}(z) \) makes the integral a principal value on \( z > 0 \):

\[
\Delta E_n^{(1)}(z \mp i0) \pm \Delta E_n^{(2,r,-)}(z) \\
\sim -i \left[ 2\pi C_n(3z)^{n-\frac{1}{2}} e^{-\frac{1}{3z}} \right] \sum_{k=0}^{N_n-1} b_n^{(k)}(3z)^k - \frac{i}{\pi} z^{-N_n}(3z)^{n-\frac{1}{2}} e^{-\frac{1}{3z}} \\
\times \text{PV} \int_0^\infty dz' \frac{z'^{-N_n}(3z')^{n+\frac{1}{2}} e^{\frac{1}{3z'}} \Delta E_n^{(2,r,-)}(z')}{z' - z}.
\]

(47)

### 3.3 Second-level hyperasymptotics

Having incorporated the dispersion relation for \( b_n(z) \) into \( \Delta E_n(z) \) via Eq. (47), we put \( \Delta E_n(z) \) [Eq. (34)] through second exponential order— with the series expansions (39), (40), and (41)— into the dispersion relation (16) and obtain the second-level hyperasymptotic expansion:

\[
E_n(g) = \sum_{j=0}^{N_n-1} \sigma_n^{(j)} + \sum_{k=0}^{N_1-1} \sigma_n^{(N_0,k)} + \sum_{i=0}^{N_2-1} \sigma_n^{(N_1,N_1,i)} + R_n(N_0, N_1, N_2, g),
\]

(48)

where the individual terms are

\[
\sigma_n^{(j)} = E_n^{(j)} g^j,
\]

(49)

\[
\sigma_n^{(N_0,k)} = -(3g)^N_0 C_n b_n^{(k)} T \left( N_0 + n + \frac{1}{2} - k, \frac{1}{3g} \right),
\]

(50)

\[
\sigma_n^{(N_0,N_1,i)} = -(3g)^N_0 C_n^2
\]

\[
\times \left[ d_n^{(1)} T^{(2)} \left( N_0 + n + \frac{1}{2} - N_1, N_1 + n + \frac{1}{2} - l, 3g \right) + c_n^{(1)} U^{(2)} \left( N_0 + n + \frac{1}{2} - N_1, N_1 + n + \frac{1}{2} - l, 3g \right) + c_n^{(1)} \pi^2 - N_0 - 2n - 1 + l T \left( N_0 + 2n + 1 - l, \frac{2}{3g} \right) \right],
\]

(51)

and where

\[
T^{(2)}(p, q, 3g) = \int_0^\infty dz(3z)^{-p} e^{-\frac{1}{3z}} \left[ \text{PV} \int_0^\infty (3z')^{-q} e^{-\frac{1}{3z'}}dz' \right],
\]

(52)

\[
U^{(2)}(p, q, 3g) = -2 \int_0^\infty dz(3z)^{-p} e^{-\frac{1}{3z}} \left[ \text{PV} \int_0^\infty (3z')^{-q} e^{-\frac{1}{3z'}} dz' \right].
\]

(53)

Evaluation of \( T \), \( T^{(2)} \), and \( U^{(2)} \) is straightforward, but will be discussed elsewhere [25]. The early \( b_n^{(k)} \) and \( c_n^{(l)} \pi^2 \) terms in effect sum the omitted RSPRT terms. The early \( d_n^{(1)} T^{(2)} \) and \( c_n^{(1)} U^{(2)} \) sum the omitted \( b_n^{(k)} \) terms. Note that \( T^{(2)}(p, q, 3g) \) differs from \( T(p, q, (3g)^{-1}) \) of Eq. (13), in that the \( \tau_1 + \tau_2 \) in the denominator in Eq. (13) is replaced by \( \tau_1 - \tau_2 \). The difference in sign indicates that a higher order Stokes phenomenon is occurring, and that singularities (and the associated exponentially subdominant contributions) which were not visible at the first stage of iteration are being uncovered at the next level. (Cf. "adjacency" in Refs. [2] and [38]).
4 Numerics for the anharmonic oscillator

For a given energy level \( n \) and anharmonicity \( g \), the errors after RSPT, hyperasymptotic, and second-level hyperasymptotic summation depend on the choice of \( N_0, (N_0, N_1) \), and \( (N_0, N_1, N_2) \). We approximate the errors by the first omitted terms in Eq. (48), asymptotic estimates of which are [25]:

\[
\begin{align*}
\langle \sigma_n^{(N_0)} \rangle_{\text{asymp}} &= -(-3g)^{N_0} C_n \Gamma(N_0 + n + \frac{1}{2}), \\
\langle \sigma_n^{(N_0,N_1)} \rangle_{\text{asymp}} &= (-3g)^{N_0} C_n^2 \Gamma(N_0 + n + \frac{1}{2} - N_1) \Gamma(N_1 + n + \frac{1}{2}) \\
&\quad \times \left[ \ln \left( N_1 + n + \frac{1}{2} \right) + \ln 12 - \psi(n + 1) \right],
\end{align*}
\]

and somewhat more crudely,

\[
\left| \langle \sigma_n^{(N_0,N_1,N_2)} \rangle_{\text{asymp}} \right| \approx \frac{9}{2} (3g)^{N_0} C_n^2 \Gamma(N_0 + n + \frac{1}{2} - N_1) \\
\times \Gamma(N_1 + n + \frac{1}{2} - N_2) \Gamma(N_2 + n + \frac{1}{2}) \\
\times \left[ \ln \left( N_2 + n + \frac{1}{2} \right) + \ln 12 - \psi(n + 1) \right]^2.
\]

The smallest \( p \)-level term occurs when (cf. [15, 17, 25]):

\[
N_{p-k} \approx \bar{N}_{p-k} = (k + 1) \left( \frac{1}{3g} - n \right), \quad (k = 0, 1, \ldots, p).
\]

At that truncation, the total number of terms is approximately

\[
\frac{1}{2} (p + 1)(p + 2) \left( \frac{1}{3g} - n \right).
\]

With these \( \bar{N}_i \), estimates of the smallest \( |\sigma_n^{(N)}| \) are (cf. [25]),

\[
\begin{align*}
\left| \langle \sigma_n^{(N_0)} \rangle_{\text{asymp}} \right| &\approx \lambda \equiv \sqrt{2\pi C_n (3g)^{-n} e^{-\frac{1}{3g}}}, \\
\left| \langle \sigma_n^{(N_0,N_1)} \rangle_{\text{asymp}} \right| &\approx \lambda^2 \left[ \ln \left( \frac{1}{3g} + \frac{1}{2} \right) + \ln 12 - \psi(n + 1) \right], \\
\left| \langle \sigma_n^{(N_0,N_1,N_2)} \rangle_{\text{asymp}} \right| &\approx \frac{9}{2} \lambda^3 \left[ \ln \left( \frac{1}{3g} + \frac{1}{2} \right) + \ln 12 - \psi(n + 1) \right]^2.
\end{align*}
\]

Predicted and numerically found smallest \( \sigma_n^{(N_0)}, \sigma_n^{(N_0,N_1)}, \) and \( \sigma_n^{(N_0,N_1,N_2)} \), reported as \( -\log_{10} |\sigma_n^{(N)}| \),

\[
p\sigma_n^{(N)} \equiv -\log_{10} |\sigma_n^{(N)}|,
\]

(which are quantitative measures of the number of significant digits expected in the partial sums) along with truncated partial sums and errors, the latter reported as the number of significant digits in the partial sum,

\[
pE^{(N)} = -\log_{10} |\text{partial sum} - \text{variational energy}|,
\]
are displayed in Table 1 for the ground \( (n = 0) \) state with anharmonicities \( g = 0.02, 0.03, 0.04, \) and 0.05. Also tabulated are the variationally calculated energies to 24 digits. All calculations were done with Mathematica. As a visual aid, the digits in the partial sums not identical with those of the variational energies have been offset by a space. The agreement with estimates is good. Improvement in accuracy from RSPT to second-level hyperasymptotics is more than two-fold but less than three-fold for the \( g \) values used here.

We plot semilogarithmically in Fig. 1 the magnitudes of the RSPT terms, the hyperterms, and second-level hyperterms for \( g = 0.02 \) for the ground state. The parameters \( N_0 \) and \( N_1 \) are taken from the "smallest-second-level-hyperterm" case given in Table 1: \( N_0 = 50, N_1 = 34, N_2 = 16 \). At each stage the terms first decrease in magnitude, reach a minimum, and then increase factorially with the order. The explanation of the sharpness of the minimum in the plot of the \( \log|\sigma^{(N_0,N_1,l)}_n| \) versus \( l \)

and that defines \( N_2 \), compared with the rounded minima for the corresponding RSPT and hyperterm plots, is that the magnitudes of the RSPT terms and hyperterms are well described by Eqs. (55) and (56), but the asymptotic \( \sigma^{(N_0,N_1,l)}_n \) undergo a single sign change "at" the minimum, near which Eq. (57) is an overestimate. The smallest regular term shown in Fig. 1 is of magnitude \( 10^{-20} \), while the largest term in the partial sum \((j = 49)\) is near \( 10^{+2} \). The cancellation of significant figures is characteristic.

5 Discussion

This paper has used dispersion relations to extend the hyperasymptotic method to sum the divergent RSPT expansion for the anharmonic oscillator energy eigenvalues through second-exponentially-small order. The exponentially small terms were derived by connecting Whittaker-based and Airy-based wave functions. Error estimates for the optimally truncated partial sums are dominated by the factor,

\[
|C_n\sqrt{2\pi(3g)^{-n}e^{-\frac{3}{2}}}|^{p+1} \left| \ln \left( \frac{1}{3g + \frac{1}{2}} \right) + \ln 12 - \psi(n + 1) \right|^p,
\]

where \( p \) denotes the highest level to which hyperasymptotics is taken. For small \( g \) the gain in accuracy is considerable.

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References


[34] F. Pham, *Multiple turning points in exact WKB analysis*, (this volume).


Figure 1: Semilogarithmic plot to show for the ground state the magnitudes of the RSPT, first-level, and second-level hyperterm contributions specified by Eqs. (49), (50) and (51). The parameter values are for the smallest-second-level-hyperterm case given in Table 1 and are $g = 0.02$, $n = 0$, $N_0 = 50$, and $N_1 = 34$. The first 50 points are the RSPT $\log_{10} |\sigma_0^{(j)}|$ vs. $j$, for $j = 0, 1, \ldots, 49$. The RSPT values continue through $j = 53$. The hyperterm $\log_{10} |\sigma_0^{(N_0,k)}|$ values start at $j = 50$, that is, $k = j - N_0 = j - 50$, and continue through $k = 52$, which is $j = 102$. The second-level hyperterm $\log_{10} |\sigma_0^{(N_0,N_1,l)}|$ values start at $j = 84$, that is, $l = j - N_0 - N_1 = j - 50 - 34$, and continue through $l = 34$ ($j = 118$). Notice the sharp dip of the smallest second-level hyperterm at $N_2 = 16$ ($j = 100$).
Table 1: Estimated (overbar) and actual smallest terms, truncated RSPT-, first-level hyperasymptotic-, and second-level hyperasymptotic- series, and error. See Eqs. (48)–(51) and (58)–(64) for definitions. [A slightly sharper estimate [25] than Eq. (58) was used for the $(\overline{N}_0, \overline{N}_1)$ of the first-level hyperasymptotic cases.]

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