J. Phys. A: Math. Gen. **35** (2002) 4017–4042 PII: S0305-4470(02)31058-8

Dispersive hyperasymptotics and the anharmonic oscillator

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Received 16 November 2001, in final form 8 February 2002
Published 26 April 2002
Online at stacks.iop.org/JPhysA/35/4017

Abstract
Hyperasymptotic summation of steepest-descent asymptotic expansions of integrals is extended to functions that satisfy a dispersion relation. We apply the method to energy eigenvalues of the anharmonic oscillator, for which there is no known integral representation, but for which there is a dispersion relation. Hyperasymptotic summation exploits the rich analytic structure underlying the asymptotics and is a practical alternative to Borel summation of the Rayleigh–Schrödinger perturbation series.

PACS numbers: 03.65.Sq, 02.30.Gp, 02.60.−x

1. Introduction

In recent years there has been renewed mathematical and physical interest in the high-order terms of asymptotic expansions of physical quantities [1–5]. Quite apart from numerical improvement, such studies lead to a better understanding of analyticity and can uncover previously unseen physical connections [3]. At the heart has been the phenomenon called *resurgence*, in which exponentially small terms, sometimes neglected in the traditional Poincaré treatment, 'rise again' in summing the high-order terms.

Traditionally, summation of asymptotic power series involves truncation before the smallest term. The residual exponentially small error is connected to the large-order behaviour of the series coefficients, which in turn is determined by the distance to the closest singularity of the Borel transform on the same Riemann sheet [6]. There are often additional (exponentially small) terms neglected by the Poincaré definition of asymptotic power series, but whose contributions may be numerically comparable and analytically essential to the reconstruction of the function that gave rise to the series. *Hyperasymptotics* refers to a sequence of approximations that transcend the truncation error of the main series while using the associated exponentially small series [1].

The possibility of exponential accuracy was noted by Stieltjes [7] and explored in particular cases by several authors [8, 9]. The idea of systematic exponential approximations, i.e. resurgence, was first conceived by Dingle [10], and later rediscovered by others, e.g. Raković and Solov'ev [11]. Its development was continued by field theorists [12] in its simplest form and in many guises, culminating in a major mathematical treatment by Écalle [13, 14]. The results are twofold: first, the divergence is seen as a consequence of associated exponentially small terms and (possibly complex) singularities and second, the procedure 'decodes' the late terms allowing a reconstruction of the function in calculable form. The former leads to complete formal expansions that include all subseries with exponentially small prefactors, regardless of size; the latter leads to a numerical procedure called hyperasymptotics [1, 2, 15–18].

The hyperasymptotic method, with sufficient information, theoretically recovers the exact function from its asymptotic expansion, and the exponentially small ambiguities characteristic of the Poincaré approach are removed. In practice, this removal stems from a careful tracking of the Stokes phenomena, whereby exponentially small terms pass in and out of the formally exact asymptotic expansion as parameters are varied.

Hyperasymptotics has been successfully applied to Laplace-type integrals [2, 15], to the solutions of certain classes and systems of ordinary differential equations [1, 16–18], and to the number-theoretic generalized Euler–Jacobi series [19]. Berry and Howls [3] and Howls and Trasler [20] have also suggested that resurgence can be applied to trace formulae associated with quantum eigenvalues, allowing the high-order terms in the quantum Weyl series to be estimated in terms of the classical periodic orbits of associated systems.

In this paper we apply hyperasymptotics to a single eigenvalue problem, the quartic one-dimensional harmonic oscillator. For each energy level there is an asymptotic power series—the usual Rayleigh–Schrödinger (RS) perturbation series—and a sequence of exponentially smaller subseries that can be derived formally by semiclassical methods. Recently, Kowalenko and Rawlinson [21] have applied the Mellin–Barnes regularization [19] to the RS power series, obtaining the first and second terms of what we will call the first-level hyperasymptotic expansion. We have developed, however, a systematic method: each successive hyper-iteration of the method consists in replacing the remainder of a truncated asymptotic expansion by a sum of terms composed of the lower-order terms of an associated asymptotic expansion multiplied by universal integrals. The connection between one series and the next has the form of a dispersion relation that arises by contour deformation of a Cauchy integral in the coupling constant plane. (We expect that these dispersion relations involve the Bender–Wu branch cuts [22–24], but we do not make the link in this paper.) In the absence of rigorous mathematical proof, we rely on heuristic arguments to justify the dispersion relations.

Although our primary interest is in the anharmonic oscillator problem, we are in fact dealing with the general problem of the derivation of hyperasymptotic approximations for functions that may not have explicit integral representations but that do satisfy a dispersion relation. In this sense, our work is analogous to that of Boyd [25], who obtained hyperasymptotic expansions for Bessel functions that can be expressed as a Stieltjes transform. We expect hyperasymptotic techniques to be widely applicable wherever dispersion relations exist [15, 25–27].

We remark that this is the first time that systematic hyperasymptotic expansions have been developed and applied numerically to an individual eigenvalue problem. A brief account was given at the 1998 Kyoto University RIMS Symposium, 'Algebraic Analysis of Singular Perturbations' [28]. We also note that the anharmonic oscillator introduces previously unseen logarithmic terms into the hyperasymptotics.

The layout of the paper is as follows: in section 2 we show how the hyperasymptotics of the Airy function, previously discussed via integral representation, follow from a dispersion relation. In section 3 we apply the dispersion relation approach to the anharmonic oscillator. The numerical results are given in section 4, and an overall discussion of our results appears in section 5. The evaluation of certain integrals is relegated to the appendix.

2. The Airy function as a prototype

Key to the invention of hyperasymptotics was an explicit integral representation for the remainder (the truncation error of the asymptotic power series). Perhaps for this reason, hyperasymptotics have been applied mainly to saddle-point expansions of functions defined by integral representations [2, 15] and to solutions of certain ordinary differential equations [1, 16–18] where integral representations of the remainders of asymptotic expansions of solutions could be derived rigorously.

In this section we show how hyperasymptotics develop from dispersion relations. Rather than jump directly to the anharmonic oscillator, we first revisit the Airy function, previously studied via integral representation. This permits us to focus on general technique without distraction from specifics peculiar to the oscillator.

The Airy function $\mathrm{Ai}(z)$ is the paradigm example because its asymptotics are relatively straightforward and uncomplicated. We rederive the hyperasymptotic expansion for $\mathrm{Ai}(z)$ given by Berry and Howls [1], but without using the explicit integral representation for $\mathrm{Ai}(z)$. In fact, only two pieces of information about the Airy function will be used: its basic asymptotic expansion (1), and its dispersion relation (11).

2.1. Asymptotic power series for the Airy function

The fundamental asymptotic expansion for $\mathrm{Ai}(z)$ is given by equation (10.4.59) of [29] (for a detailed discussion of the domains of validity, see [30]):

$$\mathrm{Ai}(z) \sim \frac{1}{2}\pi^{-1/2}z^{-1/4}\,\mathrm{e}^{-\frac{2}{3}z^{3/2}}\sum_{k=0}^{\infty}(-1)^k c_k\left(\frac{4}{3}z^{3/2}\right)^{-k} \qquad \left(|\arg z| < \frac{2}{3}\pi\right) \tag{1}$$

where

$$c_k = \frac{1}{k!}\left(\frac{1}{6}\right)_k\left(\frac{5}{6}\right)_k \tag{2}$$

and $(a)_k$ denotes the Pochhammer symbol: $(a)_0 = 1$, $(a)_k = a(a+1)\cdots(a+k-1)$. For convenience, we use a definition of c_k that is 2^k times the c_k of Abramowitz and Stegun [29], with a corresponding factor of 2 difference in the expansion variable

$$\xi = \tfrac{4}{3}z^{3/2}. \tag{3}$$

All other related asymptotic expansions for $\mathrm{Ai}(z)$ and its companion function $\mathrm{Bi}(z)$—in particular, the expansion for $\mathrm{Ai}(z)$ in the complementary sector $|\arg(-z)| < \pi/3$, and those for $\mathrm{Bi}(z)$ in the sectors $0 < \pm\arg z < 2\pi/3$ and $|\arg(-z)| < \pi/3$—are combinations of equation (1) and the continuation formulae

$$2\,\mathrm{e}^{\pm\mathrm{i}\pi/6}\mathrm{Ai}\left(\mathrm{e}^{\pm 2\pi\mathrm{i}/3}z\right) = \mathrm{Bi}(z) \pm \mathrm{i}\,\mathrm{Ai}(z). \tag{4}$$

It is convenient to focus attention on a dressed Airy function $A(\xi)$ whose asymptotic expansion is just the power series in equation (1),

$$A(\xi) \equiv 2\pi^{1/2}\left(\tfrac{3}{4}\xi\right)^{1/6}\mathrm{e}^{\xi/2}\mathrm{Ai}(z) \tag{5}$$

$$A(\xi) \sim \sum_{k=0}^{\infty}(-1)^k c_k \xi^{-k} \qquad (|\arg\xi| < \pi). \tag{6}$$

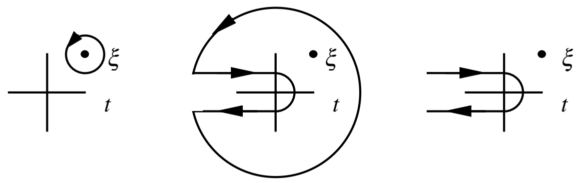

Figure 1. Contour deformation that takes the Cauchy formula (7) to the integral along the branch cut (8).

2.2. *Dispersion relation for $A(\xi)$*

We derive first the once-subtracted dispersion relation for $A(\xi)$, which is essentially a manipulation of the Cauchy integral formula

$$\frac{A(\xi)-1}{\xi}=\frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}t\,\frac{A(t)-1}{t(t-\xi)}. \tag{7}$$

Deform the integration contour as indicated in figure 1. Because of the subtraction, the contribution from the large circle vanishes as the radius increases, and the final deformed integration path wraps around the negative real axis, which is a branch cut for $A(t)$,

$$\frac{A(\xi)-1}{\xi}=\frac{1}{2\pi\mathrm{i}}\int^{(\infty\,\mathrm{e}^{+\mathrm{i}\pi},0^-,\infty\,\mathrm{e}^{-\mathrm{i}\pi})}\mathrm{d}t\,\frac{A(t)-1}{t(t-\xi)}. \tag{8}$$

Equation (8) reduces to the sum of two terms (see equation (10) below): a contribution from the origin and an integration along the negative real axis of the discontinuity in the integrand. The discontinuity of $A(t)$ across the negative t axis is provided by the continuation formulae (4)

$$A(\mathrm{e}^{\mathrm{i}\pi}\tau)-A(\mathrm{e}^{-\mathrm{i}\pi}\tau)=\mathrm{i}\,\mathrm{e}^{-\tau}A(\tau). \tag{9}$$

In this way equation (8) leads to

$$\frac{A(\xi)-1}{\xi}=-\frac{1}{\xi}+\frac{1}{2\pi}\int_0^{\infty}\mathrm{d}\tau\,\mathrm{e}^{-\tau}\,\frac{A(\tau)}{\tau(\xi+\tau)} \tag{10}$$

or, finally, to the dispersion relation

$$A(\xi)=\frac{\xi}{2\pi}\int_0^{\infty}\mathrm{d}\tau\,\frac{\mathrm{e}^{-\tau}}{\tau(\xi+\tau)}A(\tau). \tag{11}$$

Note that

$$A(\xi)=\pi^{-1/2}\xi^{1/2}\,\mathrm{e}^{\xi/2}K_{1/3}(\xi/2). \tag{12}$$

Thus equation (11) expresses $A(\xi)$ as the Stieltjes transform of a modified Bessel function, which was the starting point of Boyd's analysis [25].

2.3. *Asymptotic expansion and the dispersion relation for $A(\xi)$*

There is a close connection between the dispersion relation and the asymptotic expansion. The geometric sum

$$\frac{\xi}{\tau(\xi+\tau)}=\sum_{k=0}^{N-1}(-1)^k\frac{\tau^{k-1}}{\xi^k}+\frac{(-1)^N}{\xi^{N-1}}\frac{\tau^{N-1}}{(\xi+\tau)} \tag{13}$$

leads to what at first appears to be an alternate form of the dispersion relation (11)

$$A(\xi)=\sum_{k=0}^{N-1}(-1)^k\xi^{-k}\int_0^{\infty}\frac{\mathrm{d}\tau}{2\pi}\tau^{k-1}\,\mathrm{e}^{-\tau}A(\tau)+\frac{(-1)^N\xi^{1-N}}{2\pi}\int_0^{\infty}\mathrm{d}\tau\,\tau^{N-1}\frac{\mathrm{e}^{-\tau}}{\xi+\tau}A(\tau). \tag{14}$$

But since $A(\xi)$ has the unique asymptotic power series (6), the integrals under the sum in equation (14) must evaluate to the c_k of equation (6). That is

$$A(\xi) = \sum_{k=0}^{N-1} c_k (-1)^k \xi^{-k} + R(N, \xi) \tag{15}$$

$$R(N, \xi) = \frac{(-1)^N \xi^{1-N}}{2\pi} \int_0^\infty \mathrm{d}\tau\, \tau^{N-1} \frac{\mathrm{e}^{-\tau}}{\xi + \tau} A(\tau). \tag{16}$$

Equation (15) is exact, not asymptotic. It is in fact a family of subtracted dispersion equations—N can be any positive integer or zero. It becomes asymptotic only if the remainder term $R(N, \xi)$ is dropped. As we shall see in the next subsection, equation (15) is an engine that generates the hyperasymptotic expansions for $A(\xi)$.

2.4. *Hyperasymptotic expansions for $A(\xi)$*

The hyperasymptotic expansions for $A(\xi)$ result from substituting the dispersion relation (15) recursively into itself. The split between partial sum and remainder (i.e. the value of N) can be chosen differently at each step.

2.4.1. First-level hyperasymptotic expansion. We take $N = N_0$ in the 'target' equation (15) and $N = N_1$ in the 'substituend' equation (15) to obtain

$$A(\xi) = \sum_{k_0=0}^{N_0-1} c_{k_0} (-1)^{k_0} \xi^{-k_0} + \frac{(-1)^{N_0}}{2\pi \xi^{N_0}} \sum_{k_1=0}^{N_1-1} c_{k_1} (-1)^{k_1} T(N_0 - k_1, \xi) + R(N_0, N_1, \xi) \tag{17}$$

where the first-level T functions, which depend on $N_0 - k_1$ and ξ, but are independent of $A(\xi)$, can be expressed in terms of the gamma and incomplete gamma functions

$$T(N_0 - k_1, \xi) \equiv \xi \int_0^\infty \mathrm{d}\tau_1 \frac{\tau_1^{N_0-k_1-1}\, \mathrm{e}^{-\tau_1}}{\xi + \tau_1} \tag{18}$$

$$= \xi^{N_0-k_1}\, \mathrm{e}^{\xi}\, \Gamma(N_0 - k_1)\Gamma(1 - N_0 + k_1, \xi) \tag{19}$$

and where the first-level remainder is a double integral with $A(\tau_2)$ in the integrand

$$R(N_0, N_1, \xi) = \frac{(-1)^{N_0+N_1}}{(2\pi)^2 \xi^{N_0-1}} \int_0^\infty \mathrm{d}\tau_1\, \tau_1^{N_0-N_1} \frac{\mathrm{e}^{-\tau_1}}{\xi + \tau_1} \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{N_1-1} \frac{\mathrm{e}^{-\tau_2}}{\tau_1 + \tau_2} A(\tau_2). \tag{20}$$

2.4.2. Second-level hyperasymptotic expansion. Substitution of the dispersion relation (15) into the remainder term of equation (17) gives the second-level hyperasymptotic expansion

$$\begin{aligned} A(\xi) = {} & \sum_{k_0=0}^{N_0-1} c_{k_0} (-1)^{k_0} \xi^{-k_0} + \frac{(-1)^{N_0}}{2\pi \xi^{N_0}} \sum_{k_1=0}^{N_1-1} c_{k_1} (-1)^{k_1} T(N_0 - k_1, \xi) + \frac{(-1)^{N_0+N_1}}{(2\pi)^2 \xi^{N_0}} \\ & \times \sum_{k_2=0}^{N_2-1} c_{k_2} (-1)^{k_2} T(N_0 - N_1, N_1 - k_2, \xi) + R(N_0, N_1, N_2, \xi) \end{aligned} \tag{21}$$

where the second-level T functions (again independent of $A(\xi)$) are defined by

$$T(N_0 - N_1, N_1 - k_2, \xi) \equiv \xi \int_0^\infty \mathrm{d}\tau_1\, \tau_1^{N_0-N_1} \frac{\mathrm{e}^{-\tau_1}}{\xi + \tau_1} \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{N_1-k_2-1} \frac{\mathrm{e}^{-\tau_2}}{\tau_1 + \tau_2} \tag{22}$$

and where the second-level remainder is the triple-integral

$$R(N_0, N_1, N_2, \xi) = \frac{(-1)^{N_0+N_1+N_2}}{(2\pi)^3 \xi^{N_0-1}} \int_0^\infty \mathrm{d}\tau_1\, \tau_1^{N_0-N_1} \frac{\mathrm{e}^{-\tau_1}}{\xi + \tau_1} \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{N_1-N_2} \frac{\mathrm{e}^{-\tau_2}}{\tau_1 + \tau_2} \times \int_0^\infty \mathrm{d}\tau_3\, \tau_3^{N_2-1} \frac{\mathrm{e}^{-\tau_3}}{\tau_2 + \tau_3} A(\tau_3). \tag{23}$$

2.4.3. Higher-level hyperasymptotic expansions. The next higher-level hyperasymptotic expansion follows by substituting equation (15) into the remainder term of equation (21), and so forth. In this manner, dispersion relations generate hyperasymptotic expansions to successively higher orders.

2.4.4. Comments. The only explicit information about $A(\xi)$ in the hyperasymptotic expansions (17) and (21) comes in through the series coefficients c_k: the T factors are of a universal form.

The numerical utility of the first-level hyperasymptotic expansion (17) is that the remainder term $R(N_0, N_1, \xi)$ can often be made orders of magnitude smaller than the remainder term $R(N_0, \xi)$ for the standard asymptotic expansion (15) by appropriate choice of N_0 and N_1. The lower-order c_{k_1} in effect are used to sum the omitted higher-order asymptotic terms c_{k_0} with $k_0 \geqslant N_0$.

Similarly, the numerical utility of the second-level hyperasymptotic expansion (21) is that the remainder term $R(N_0, N_1, N_2, \xi)$ can often be made orders of magnitude smaller than the remainder term $R(N_0, N_1, \xi)$ for the hyperasymptotic expansion (17) by appropriate choice of N_0, N_1 and N_2. The lower-order c_{k_2} in effect are used to sum the omitted late first-level hyperasymptotic terms c_{k_1} with $k_1 \geqslant N_1$.

Therefore, the advantage of the hyperasymptotic method is that successively higher computational accuracy is extracted from the divergent power-series coefficients c_k. The disadvantage is that the T's of the successively higher-level series become increasingly more cumbersome to calculate (for methods, see [1, 18, 26, 27]).

Finally we mention that there are other new techniques for summing divergent series, e.g. Weniger [31–34].

2.5. Beyond Airy functions

The Airy function is particularly simple in the sense that it appears in its own dispersion relation. More typically the function on the right in a dispersion relation differs from that on the left. Equation (15) might be replaced by a sequence of equations such as

$$A_i(\xi) = \sum_{k=0}^{N-1} c_{i,k} (-1)^k \xi^{-k} + \frac{(-1)^N \xi^{1-N}}{2\pi} \int_0^\infty \mathrm{d}\tau\, f_{i+1}(\tau) \tau^{N-1} \frac{\mathrm{e}^{-\tau}}{\xi + \tau} A_{i+1}(\tau) \tag{24}$$

or perhaps

$$A_i(\xi) = \sum_{k=0}^{N-1} c_{i,k} \xi^{-k} + \frac{\xi^{1-N}}{2\pi} \int_0^\infty \mathrm{d}\tau\, f_{i+1}(\tau) \tau^{N-1} \frac{\mathrm{e}^{-\tau}}{\xi - \tau} A_{i+1}(\tau) \tag{25}$$

with a pole in the integrand on the positive axis. Incidentally, a pole is equivalent to a Stokes phenomenon [1]. It may be thought of as arising from the Borel summation of terms with equal phase, and in such a case, information about the branch of $A_i(\xi)$ is required to take explicit account of the pole by suitable deformation of the integration contour [1]. A linear combination of equations such as (24) and (25) is also possible.

In equations (24) and (25) the $A_i(\xi)$ denote functions having asymptotic power series that start with constant terms. The modifying function $f_{i+1}(\tau)$ could be a constant times a power

$$f_{i+1}(\tau) = a_{i+1}\tau^{\sigma_{i+1}} \tag{26}$$

or a constant times a power times a logarithm

$$f_{i+1}(\tau) = a_{i+1}\tau^{\sigma_{i+1}} \ln \tau \tag{27}$$

or something more complicated. Previous applications of hyperasymptotics have not explicitly encountered the logarithmic scenario (27).

The great advantage of the Airy function is that it permits development of the hyperasymptotic aspects without having to discuss the asymptotics of the functions appearing on the right-hand side of the dispersion formula, since for the Airy function they are the same. This is not the case for the anharmonic oscillator eigenvalues, where all the scenarios represented by equations (24)–(27) occur.

3. The anharmonic oscillator

The quartic anharmonic oscillator is characterized by the Schrödinger equation

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2 + gx^4 - E_n(g)\right]\psi(x) = 0. \tag{28}$$

The nth perturbed energy eigenvalue $E_n(g)$ has a factorially divergent but Borel-summable [35] asymptotic power series given by RS perturbation theory

$$E_n(g) \sim E_n^{\mathrm{RS}}(g) \equiv \sum_{j=0}^{\infty} E_n^{(j)} g^j \qquad (|\arg g| < \pi). \tag{29}$$

As a function of the anharmonicity constant g, the energy has a cut on the negative real g axis, and the following once-subtracted dispersion relation [23, 36] holds

$$E_n(g) = E_n^{(0)} + \frac{(-g)}{2\pi\mathrm{i}}\int_0^\infty \frac{\Delta E_n(z)}{z(z+g)}\,\mathrm{d}z \tag{30}$$

where $E_n^{(0)} = n + 1/2$, and where

$$\Delta E_n(z) \equiv E_n(\mathrm{e}^{-\mathrm{i}\pi}z) - E_n(\mathrm{e}^{+\mathrm{i}\pi}z) \tag{31}$$

denotes the discontinuity of $E_n(g)$ across the negative real g axis when $g = -z$. The combination of factorial divergence and dispersion relation suggests the possibility of hyperasymptotics.

The dispersion relation (30) leads to a formal equation for partial summation of the RS series plus remainder, the analogue of equations (15) and (24)

$$E_n(g) = \sum_{j=0}^{N_0-1} E_n^{(j)} g^j + R_n(N_0, g) \tag{32}$$

$$R_n(N_0, g) = \frac{(-1)^{N_0} g^{N_0}}{2\pi\mathrm{i}}\int_0^\infty \frac{z^{-N_0}}{z+g}\Delta E_n(z)\,\mathrm{d}z. \tag{33}$$

3.1. Energy discontinuity across the negative g axis to first-exponentially-small order; first-level hyperasymptotic expansion

The discontinuity of $E_n(g)$ across the negative g axis and its relation to the asymptotics of the RS series were discussed previously in [37] where the following first-exponentially-small-order asymptotic expansion was derived,

$$\Delta E_n(z) \sim \Delta E_n^{\{1\}} = -2\pi \mathrm{i} C_n (3z)^{-n-\frac{1}{2}} \, \mathrm{e}^{-\frac{1}{3z}} b_n^{\mathrm{series}}(z) \tag{34}$$

where

$$C_n = \frac{12^{n+\frac{1}{2}}}{\pi\sqrt{2\pi}\,\Gamma(n+1)} \tag{35}$$

$$b_n^{\mathrm{series}}(z) \equiv \sum_{k=0}^{\infty} b_n^{(k)} (3z)^k. \tag{36}$$

The coefficients $b_n^{(k)}$ turn out to be polynomials of degree $2k$ in the quantum number n, with $b_n^{(0)} = 1$. Numerical values through 50th order were previously tabulated for the first three states ($n = 0, 1, 2$) in [37]. Note that $3g$ here corresponds to ξ^{-1} in the Airy function discussion, and similarly $3z$ corresponds to τ^{-1}.

The expansion (34)–(36) permits us to calculate the first-level hyperasymptotic contribution to the remainder after RS summation through order $N_0 - 1$

$$R_n(N_0, g) = -(-3g)^{N_0} C_n \sum_{k=0}^{N_1-1} b_n^{(k)} T\left(N_0 + n + \frac{1}{2} - k, \frac{1}{3g}\right) + R_n(N_0, N_1, g) \tag{37}$$

where T is the same function (18) that appeared in the Airy function hyperasymptotics, and which follows from the substitution $3z = 1/\tau$,

$$T\left(N_0 + n + \frac{1}{2} - k, \frac{1}{3g}\right) \equiv \int_0^{\infty} \frac{(3z)^{-N_0-n-\frac{1}{2}+k} \, \mathrm{e}^{-\frac{1}{3z}}}{z+g} \, \mathrm{d}z. \tag{38}$$

3.2. Heuristic derivation of a dispersion relation for $b_n(z)$

In the companion paper [38] we have extended the method of [37] to calculate the energy discontinuity to second-exponentially-small order

$$\Delta E_n(z) \sim \Delta E_n^{\{1\}} + \Delta E_n^{\{2\}}. \tag{39}$$

Does second-level hyperasymptotics follow from $\Delta E_n^{\{2\}}$ in the same way that first-level hyperasymptotics follows from $\Delta E_n^{\{1\}}$? The answer is yes (it follows) and no (not exactly in the same way). First, there is a direct, second-exponentially-small contribution to $\Delta E_n(z)$ that makes a second-level hyperasymptotic contribution via equations (32) and (33). Second, one must sum the series $b_n^{\mathrm{series}}(z)$ that appears as a factor in $\Delta E_n^{\{1\}}$,

$$b_n(z) = \text{Borel sum of } \sum_{k=0}^{\infty} b_n^{(k)} (3z)^k \tag{40}$$

which would require a dispersion relation that in turn should involve the discontinuity of $b_n(z)$, namely, the discontinuity of the discontinuity of $E_n(g)$. How to extract the relevant information from $\Delta E_n^{\{2\}}$, which is a part of the (noniterated) discontinuity of $E_n(g)$ in second-exponential-order, is not immediately obvious.

We adopt an exploratory attitude and proceed intuitively, but nonrigorously. First note that $E_n(g)$ satisfies the Schwarz reflection principle, whose consequence is that $\Delta E_n(z)$ is purely imaginary for z real and positive. Further, the analyticity of $E_n(g)$ implies that $\Delta E_n(z)$ is analytic and *a fortiori* continuous for positive z.

The first-exponentially-small expansion $\Delta E_n^{\{1\}}$ is $(-\mathrm{i})$ times a real prefactor times the formally real power series $b_n^{\mathrm{series}}(z)$, when z is positive, and at first glance would seem to satisfy the Schwarz principle. Closer inspection reveals a problem. Since the late coefficients $b_n^{(k)}$ are all negative and grow slightly faster than factorially with k [37, 38], we infer that $b_n(z)$ has a cut on the positive z axis, and that the discontinuity across the cut is purely imaginary and exponentially small, which after multiplication by the appropriate exponentially small prefactor becomes a second-exponentially-small-order real discontinuity in the Borel sum of $\Delta E_n^{\{1\}}(z)$, seemingly in conflict with the Schwarz principle and the continuity of $\Delta E_n(z)$.

The conflict is resolved by $\Delta E_n^{\{2\}}$, which has the form [38]

$$\Delta E_n^{\{2\}}(z) = \Delta E_n^{\{2,r\}}(z) + \mathrm{i}\Delta E_n^{\{2,i\}}(z) \tag{41}$$

where for real z both $\Delta E_n^{\{2,r\}}(z)$ and $\Delta E_n^{\{2,i\}}(z)$ are formally real expansions; however, $\Delta E_n^{\{2,r\}}(z)$ changes sign discontinuously on $z > 0$

$$\Delta E_n^{\{2,r\}}(z - \mathrm{i}0) = -\Delta E_n^{\{2,r\}}(z + \mathrm{i}0) \qquad (z > 0). \tag{42}$$

The explicit $\Delta E_n^{\{2,r\}}(z)$ series must cancel in second exponential order the implicit discontinuity of the Borel sum of the $\Delta E_n^{\{1\}}$ series, because $\Delta E_n(z)$ must be imaginary and continuous on $z > 0$. Symbolically, for $z > 0$

$$\begin{aligned}\Delta E_n^{\{1\}}(z \pm \mathrm{i}\varepsilon) &+ \Delta E_n^{\{2,r\}}(z \pm \mathrm{i}\varepsilon) + \mathrm{i}\Delta E_n^{\{2,i\}}(z \pm \mathrm{i}\varepsilon) \\ &\xrightarrow{\text{Borel sum}} -2\pi \mathrm{i} C_n (3z)^{-n-\frac{1}{2}}\, \mathrm{e}^{-\frac{1}{3z}}\, \mathrm{Re}[b_n(z)] + \mathrm{i}\Delta E_n^{\{2,i\}}(z) + \mathrm{O}(\mathrm{e}^{-\frac{1}{3z}})^3.\end{aligned} \tag{43}$$

The cancellation itself is

$$2\pi C_n (3z)^{-n-\frac{1}{2}}\, \mathrm{e}^{-\frac{1}{3z}}\, \mathrm{Im}[b_n(z \pm \mathrm{i}\varepsilon)] + \Delta E_n^{\{2,r\}}(z \pm \mathrm{i}\varepsilon) = 0 + \mathrm{O}(\mathrm{e}^{-\frac{1}{3z}})^3. \tag{44}$$

Note that it is necessary to specify whether the real axis is approached from above or below, because $b_n(z)$ and $\Delta E_n^{\{2,r\}}(z)$ are discontinuous on $z > 0$; hence the $\pm \mathrm{i}\varepsilon$.

We elaborate on this last equation to obtain an asymptotic expansion for the discontinuity $\Delta b_n(z)$ in $b_n(z)$. First take z real and positive, and define

$$\Delta b_n(z) \equiv b_n(z - \mathrm{i}0) - b_n(z + \mathrm{i}0) \qquad (z > 0). \tag{45}$$

For z complex, define $\Delta b_n(z)$ by analytic continuation from $z > 0$. To avoid the sign confusion that could arise when $\Delta E_n^{\{2,r\}}(z \pm \mathrm{i}0)$ appears in a dispersion relation, in [38] we introduced the symbol $\Delta E_n^{\{2,r,-\}}(z)$ to denote an expansion that is *formally continuous* across the real axis, that coincides with $\Delta E_n^{\{2,r\}}(z)$ when $\mathrm{Im}\, z < 0$, but that is $-\Delta E_n^{\{2,r\}}(z)$ when $\mathrm{Im}\, z > 0$. The resulting asymptotic expansion for $\Delta b_n(z)$ is then

$$\Delta b_n(z) \sim \frac{-2\mathrm{i}}{2\pi C_n} (3z)^{n+\frac{1}{2}}\, \mathrm{e}^{\frac{1}{3z}}\, \Delta E_n^{\{2,r,-\}}(z). \tag{46}$$

That the formally real series $\Delta E_n^{\{2,r,-\}}(z)$ yields a formula for the discontinuity of the Borel sum of the $\Delta E_n^{\{1\}}(z)$, and thereby provides a dispersion relation, was conjectured in essence by Damburg and Propin [39] in the context of the separation constants for hydrogen in an external electric field, for which the separated equations are equivalent to a radially symmetric two-dimensional anharmonic oscillator.

A formal statement of our conjecture is that

$$b_n(z) = \frac{-1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{d}\zeta\,\frac{\Delta b_n(\zeta)}{\zeta - z} \tag{47}$$

$$= \sum_{k=0}^{N_1-1} b_n^{(k)}(3z)^k - \frac{z^{N_1}}{2\pi\mathrm{i}}\int_0^\infty \mathrm{d}\zeta\,\frac{\zeta^{-N_1}\Delta b_n(\zeta)}{\zeta - z} \tag{48}$$

$$\sim \sum_{k=0}^{N_1-1} b_n^{(k)}(3z)^k + \frac{2z^{N_1}}{(2\pi)^2 C_n}\int_0^\infty \mathrm{d}\zeta\,\frac{\zeta^{-N_1}(3\zeta)^{n+\frac{1}{2}}\,\mathrm{e}^{\frac{1}{3\zeta}}\,\Delta E_n^{\{2,r,-\}}(\zeta)}{\zeta - z}. \tag{49}$$

The explicit expansion for $\Delta E_n^{\{2,r,-\}}(z)$ derived in [38] leads to

$$\Delta b_n(z) \sim 2\pi\mathrm{i}C_n(3z)^{-n-\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{3z}}\left\{\sum_{l=1}^{\infty} d_n^{(l)}(3z)^l - 2\left[\ln\left(\frac{z}{4}\right) + \psi(n+1)\right]\sum_{l=0}^{\infty} c_n^{(l)}(3z)^l\right\} \tag{50}$$

where the expansion coefficients $c_n^{(l)}$ and $d_n^{(l)}$ are defined in [38]. The formula for $\Delta E_n^{\{2,i\}}(z)$, which involves the same expansion coefficients $c_n^{(l)}$, is [38]

$$\Delta E_n^{\{2,i\}}(z) = \frac{\pi}{2}\left[2\pi C_n(3z)^{-n-\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{3z}}\right]^2 \sum_{l=0}^{\infty} c_n^{(l)}(3z)^l. \tag{51}$$

3.3. Second-level hyperasymptotic expansion

Armed with the dispersion relation for $b_n(z)$, we follow again the example of the Airy function to obtain second-level hyperasymptotics, but the structure is more complicated. There are two separate pieces: the first is from $\mathrm{i}\Delta E_n^{\{2,i\}}(z)$; the second is from $\Delta E_n^{\{2,r\}}(z)$ and sums the $b_n(z)$ series in first exponential order

$$R_n(N_0, N_1, g) = \frac{(-1)^{N_0} g^{N_0}}{2\pi}\int_0^\infty \frac{z^{-N_0}}{z+g}\left[\Delta E_n^{\{2,i\}}(z) + \cdots\right]\mathrm{d}z + C_n(-g)^{N_0}$$
$$\times \int_0^\infty \frac{z^{-N_0}}{z+g}(3z)^{-n-\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{3z}}\left[z^{N_1}\frac{1}{2\pi\mathrm{i}}\int_0^\infty \frac{\zeta^{-N_1}\Delta b_n(\zeta)}{\zeta - z}\,\mathrm{d}\zeta\right]\mathrm{d}z. \tag{52}$$

The contribution from the first piece is similar to that of $\Delta E_n^{\{1\}}$ in equations (33)–(38)

$$\frac{(-1)^{N_0} g^{N_0}}{2\pi}\int_0^\infty \frac{z^{-N_0}}{z+g}\Delta E_n^{\{2,i\}}(z)\,\mathrm{d}z$$
$$= (-3g)^{N_0} C_n^2 \sum_{l=0}^{\infty} c_n^{(l)}\pi^2 2^{-N_0-2n-1+l}\,T\left(N_0 + 2n + 1 - l, \frac{2}{3g}\right). \tag{53}$$

The contribution of the second piece is analogous to equations (24) and (25) combined sequentially

$$C_n(-g)^{N_0}\int_0^\infty \frac{z^{-N_0}}{z+g}(3z)^{-n-\frac{1}{2}}\mathrm{e}^{-\frac{1}{3z}}\left[z^{N_1}\frac{1}{2\pi\mathrm{i}}\int_0^\infty \frac{\zeta^{-N_1}\Delta b_n(\zeta)}{\zeta - z}\,\mathrm{d}\zeta\right]\mathrm{d}z \tag{54}$$

$$= C_n^2(-g)^{N_0}\int_0^\infty \frac{z^{-N_0}}{z+g}(3z)^{-n-\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{3z}}\left[z^{N_1}\int_0^\infty (3\zeta)^{-n-\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{3\zeta}}\,\frac{\zeta^{-N_1}}{\zeta - z}\,\mathrm{d}\zeta\right]$$
$$\times \left\{\sum_{l=1}^{\infty} d_n^{(l)}(3\zeta)^l - 2\left[\ln\left(\frac{\zeta}{4}\right) + \psi(n+1)\right]\sum_{l=0}^{\infty} c_n^{(l)}(3\zeta)^l\right\}\mathrm{d}z. \tag{55}$$

We are thus led to the second-level hyperasymptotic formula for $R_n(N_0, N_1, g)$, the remainder in equation (37)

$$
\begin{aligned}
R_n(N_0, N_1, g) = (-3g)^{N_0} C_n^2 \Bigg[& \sum_{l=1}^{N_2-1} d_n^{(l)} T^{\{2\}} \left(N_0 + n + \frac{1}{2} - N_1, N_1 + n + \frac{1}{2} - l, 3g \right) \\
& + \sum_{l=0}^{N_2-1} c_n^{(l)} U^{\{2\}} \left(N_0 + n + \frac{1}{2} - N_1, N_1 + n + \frac{1}{2} - l, 3g \right) \\
& + \sum_{l=0}^{N_2-1} c_n^{(l)} \pi^2 2^{-N_0-2n-1+l} T \left(N_0 + 2n + 1 - l, \frac{2}{3g} \right) \Bigg] \\
& + R_n(N_0, N_1, N_2, g)
\end{aligned} \tag{56}
$$

where

$$
T^{\{2\}}(p, q, 3g) = \int_0^\infty (3z)^{-p} \frac{\mathrm{e}^{-\frac{1}{3z}}}{z+g} \left[\int_0^\infty (3\zeta)^{-q} \frac{\mathrm{e}^{-\frac{1}{3\zeta}}}{\zeta - z} \, \mathrm{d}\zeta \right] \mathrm{d}z \tag{57}
$$

$$
U^{\{2\}}(p, q, 3g) = -2 \int_0^\infty (3z)^{-p} \frac{\mathrm{e}^{-\frac{1}{3z}}}{z+g} \int_0^\infty (3\zeta)^{-q} \frac{\mathrm{e}^{-\frac{1}{3z}}}{\zeta - z} \left[\ln\left(\frac{\zeta}{4}\right) + \psi(n+1) \right] \mathrm{d}\zeta \, \mathrm{d}z. \tag{58}
$$

We remark that $T^{\{2\}}(p, q, 3g)$ differs from the $T(p, q, (3g)^{-1})$ of equation (22) in that the $\tau_1 + \tau_2$ in the denominator in equation (22) is replaced by $\tau_1 - \tau_2$. The difference in sign indicates that a higher-order Stokes phenomenon is occurring, and that singularities (and the associated exponentially subdominant contributions) which were not visible at the first stage of iteration are being uncovered at the next level (cf 'adjacency' in [2] and [40]). The evaluation of $T^{\{2\}}$ and $U^{\{2\}}$ is discussed in the appendix, where the singularity in the denominator of the integrand and the logarithmic factor are treated in detail.

4. Numerical calculations

The picture of how hyperasymptotics works would be incomplete without a numerical example. In this section we present calculations for the anharmonic oscillator ground and first excited states.

The formula for the energy through second hyperasymptotic order is a combination of equations (32), (37) and (56),

$$
\begin{aligned}
E_n(g) = \sum_{j=0}^{N_0-1} E_n^{(j)} g^j & - (-3g)^{N_0} C_n \sum_{k=0}^{N_1-1} b_n^{(k)} T \left(N_0 + n + \frac{1}{2} - k, \frac{1}{3g} \right) \\
& + (-3g)^{N_0} C_n^2 \Bigg[\sum_{l=1}^{N_2-1} d_n^{(l)} T^{\{2\}} \left(N_0 + n + \frac{1}{2} - N_1, N_1 + n + \frac{1}{2} - l, 3g \right) \\
& + \sum_{l=0}^{N_2-1} c_n^{(l)} U^{\{2\}} \left(N_0 + n + \frac{1}{2} - N_1, N_1 + n + \frac{1}{2} - l, 3g \right) \\
& + \sum_{l=0}^{N_2-1} c_n^{(l)} \pi^2 2^{-N_0-2n-1+l} T \left(N_0 + 2n + 1 - l, \frac{2}{3g} \right) \Bigg] \\
& + R_n(N_0, N_1, N_2, g)
\end{aligned} \tag{59}
$$

Table 1. Variational energies for the ground and first excited states of the quartic anharmonic oscillator as a function of the anharmonicity constant g.

g	$E_0(g)$	$E_1(g)$
0.02	0.514 086 427 318 015 764 724 637	1.568 239 676 804 948 278 327 021
0.03	0.520 561 719 873 001 952 996 419	1.598 456 089 397 955 550 305 114
0.04	0.526 733 964 393 435 586 948 803	1.626 748 400 185 579 933 423 763
0.05	0.532 642 754 771 858 844 428 546	1.653 436 006 576 456 753 564 061

where C_n is the numerical constant given by equation (35), $E_n^{(j)}$ are the RS energy coefficients, and the $b_n^{(k)}$, $c_n^{(k)}$ and $d_n^{(k)}$ coefficients are obtained in [38].

4.1. Optimum choice of $N_0, N_1, \ldots, N_p$

The values of $N_0, N_1, \ldots, N_p$ that give best numerical accuracy can be predicted theoretically by estimating the remainder term $R_n(N_0, N_1, \ldots, N_p, g)$ as the size of the first neglected term; the estimate is then globally minimized with respect to $N_0, \ldots,$ and N_p. This approach is similar to the more rigorous method of Olde Daalhuis and Olver [16, 18]. The leading-order growth of the RS coefficients $E_n^{(j)}$ and hyperasymptotic $b_n^{(k)}$, $c_n^{(k)}$ and $d_n^{(k)}$ coefficients [37, 38] is of the same class as for the expansions of Olde Daalhuis and Olver, the commonly encountered factorial times a power—albeit modified here by a slowly varying logarithm. Thus one expects the truncation algorithm that minimizes the remainder to be similar. Their result in our notation is

$$N_k \approx (k+1)\left(\frac{1}{3g} - n\right) \qquad (k = 0, 1, \ldots, p). \tag{60}$$

For a given energy level n and anharmonicity constant g, the estimated truncation constant N_p of the exponentially *smallest* order is independent of p. The total number of terms after p stages of hyperasymptotics (which naturally does depend on p) is approximately given by

$$\frac{1}{2}(p+1)(p+2)\left(\frac{1}{3g} - n\right) \tag{61}$$

and thus increases quadratically with the highest hyperasymptotic level. We shall see that these predicted truncations are supported by explicit calculations.

4.2. Numerical illustrations

For simplicity we consider the ground and first excited states with anharmonicities $g = 0.02$, 0.03, 0.04 and 0.05. (Due to the expressions derived in the companion paper [38] for the coefficients $b_n^{(k)}$, $c_n^{(k)}$ and $d_n^{(k)}$, higher excited states are computationally similar in cost to calculate, but, because of equation (60), the method in its present form is useful in the smaller range of g given by $1/(3g) > n$.) The variational energies are given to 24 digits in table 1. In subsequent tables we shall compare these variational energies with RS partial sums and first- and second-level hyperasymptotic partial sums.

4.2.1. RS partial sum. The RS terms ultimately alternate in sign. As N_0 increases while g is small and fixed, the terms generally get smaller in magnitude, reach a minimum, and then

Table 2. Notation for first omitted terms and partial sums of the RS, first-level hyperasymptotic and second-level hyperasymptotic series to specify the numerical calculations of section 4.

$$\sigma_n^{(N_0)} = E_n^{(N_0)} g^{N_0}$$

$$\sigma_n^{(N_0,N_1)} = (-3g)^{N_0} C_n b_n^{(N_1)} T\left(N_0 + n + \tfrac{1}{2} - N_1, \tfrac{1}{3g}\right)$$

$$\begin{aligned}\sigma_n^{(N_0,N_1,N_2)} = (-3g)^{N_0} C_n^2 \Big[& d_n^{(N_2)} T^{\{2\}}\left(N_0 + n + \tfrac{1}{2} - N_1, N_1 + n + \tfrac{1}{2} - N_2, 3g\right) \\ & + c_n^{(N_2)} U^{\{2\}}\left(N_0 + n + \tfrac{1}{2} - N_1, N_1 + n + \tfrac{1}{2} - N_2, 3g\right) \\ & + c_n^{(N_2)} \pi^2 2^{-N_0-2n-1+N_2} T\left(N_0 + 2n + 1 - N_2, \tfrac{2}{3g}\right)\Big]\end{aligned}$$

$$\Sigma_n^{(N_0)} = \sum_{k=0}^{N_0-1} E_n^{(k)} g^k$$

$$\Sigma_n^{(N_0,N_1)} = \sum_{k=0}^{N_0-1} E_n^{(k)} g^k + (-3g)^{N_0} C_n \sum_{k=0}^{N_1-1} b_n^{(k)} T\left(N_0 + n + \frac{1}{2} - k, \frac{1}{3g}\right)$$

$$\begin{aligned}\Sigma_n^{(N_0,N_1,N_2)} = & \sum_{k=0}^{N_0-1} E_n^{(k)} g^k + (-3g)^{N_0} C_n \sum_{k=0}^{N_1-1} b_n^{(k)} T\left(N_0 + n + \frac{1}{2} - k, \frac{1}{3g}\right) \\ & + (-3g)^{N_0} C_n^2 \sum_{k=0}^{N_2-1} \Big[d_n^{(k)} T^{\{2\}}\left(N_0 + n + \tfrac{1}{2} - N_1, N_1 + n + \tfrac{1}{2} - k, 3g\right) \\ & + c_n^{(k)} U^{\{2\}}\left(N_0 + n + \tfrac{1}{2} - N_1, N_1 + n + \tfrac{1}{2} - k, 3g\right) \\ & + c_n^{(k)} \pi^2 2^{-N_0-2n-1+k} T\left(N_0 + 2n + 1 - k, \tfrac{2}{3g}\right)\Big]\end{aligned}$$

Table 3. Smallest RS term and corresponding N_0: asymptotic estimate $\sigma_{n,\mathrm{as}}^{(\bar{N}_0)}$ and actual $\sigma_n^{(N_0)}$ (see table 2 for notation).

		Asymptotic estimate		Actual	
n	g	$\bar{N}_0$	$\sigma_{n,\mathrm{as}}^{(\bar{N}_0)}$	N_0	$\sigma_n^{(N_0)}$
0	0.02	16.7	6.4×10^{-8}	17	5.8×10^{-8}
	0.03	11.1	1.6×10^{-5}	11	1.4×10^{-5}
	0.04	8.3	2.7×10^{-4}	8	-2.1×10^{-4}
	0.05	6.7	1.4×10^{-3}	6	-1.0×10^{-3}
1	0.02	15.7	12.7×10^{-6}	15	8.4×10^{-6}
	0.03	10.1	2.2×10^{-3}	9	1.0×10^{-3}
	0.04	7.3	26.5×10^{-3}	6	-8.1×10^{-3}
	0.05	5.7	11.2×10^{-2}	4	-2.5×10^{-2}

increase factorially fast. Following the notation of table 2, we denote the N_0th term of the RS expansion by $\sigma_n^{(N_0)}$ and its leading asymptotic form by $\sigma_{n,\mathrm{as}}^{(N_0)}$

$$\sigma_{n,\mathrm{as}}^{(N_0)} = -C_n(-3g)^{N_0}\Gamma\left(N_0 + n + \tfrac{1}{2}\right). \tag{62}$$

With Stirling's approximation, the $\sigma_{n,\mathrm{as}}^{(N_0)}$ has minimum magnitude when N_0 is given by

$$N_0 \approx \bar{N}_0 \equiv \frac{1}{3g} - n \tag{63}$$

for which

$$\left|\sigma_{n,\mathrm{as}}^{(\bar{N}_0)}\right| \approx \sqrt{2\pi} C_n (3g)^{-n} \,\mathrm{e}^{-\frac{1}{3g}}. \tag{64}$$

Table 3 compares the asymptotic estimates of equations (63) and (64) with the numerically computed smallest terms for each value of g and n. In table 4 we display the partial sums of the RS series, truncated just before the smallest term—the standard approach to summing

Table 4. RS energies and accuracy (as number of significant digits) for the ground and first excited states (see table 2 for notation). The 'error' in the last column denotes the difference $E_n(g) - \Sigma_n^{(N_0)}$ between the variational eigenvalue and the corresponding partial sum. As a visual aid, the digits in the partial sums not identical with those of the variational energies have been offset by a space.

n	g	N_0	$\Sigma_n^{(N_0)}$	$-\log\left\|\sigma_n^{(N_0)}\right\|$	$-\log\|\text{error}\|$
0	0.02	17	0.514086 399	7.2	7.5
	0.03	11	0.5205 548	4.9	5.2
	0.04	8	0.526 837	3.7	4.0
	0.05	6	0.53 315	3.0	3.3
1	0.02	15	1.56823 55	5.1	5.4
	0.03	9	1.59 796	3.0	3.3
	0.04	6	1.6 308	2.1	2.4
	0.05	4	1.6 665	1.6	1.9

an asymptotic series without hyperasymptotics. Also tabulated is a measure of the number of significant digits, expected from the size of the first omitted term, $-\log\left|\sigma_n^{(N_0)}\right|$, and found as the variational energy minus truncated series, $-\log\left|E_n(g) - \Sigma_n^{(N_0)}\right|$.

The error in the truncated series is slightly smaller than the magnitude of the first omitted term because the RS terms oscillate in sign. The index N_0 of the smallest term is smaller than predicted for the $n = 1$ case, especially for the larger values of g, for which the predicted N_0 is already small, because the leading asymptotic term overestimates the magnitude of $E_1^{(N_0)}$ when N_0 is small. (The leading asymptotic term estimates $E_0^{(N_0)}$ better than $E_1^{(N_0)}$ for the same value of N_0.)

4.2.2. Smallest hyperterm and hyperterm partial sum. The asymptotic form for small g of the hyperterm contribution $\sigma_n^{(N_0,N_1)}$ follows from equation (85) in the companion paper [38] and either (A.3) or (A.5) below, depending on whether $N_0 + n - k \ll (3g)^{-1}$ or $N_0 + n - k \approx (3g)^{-1}$, the difference being a factor of $\frac{1}{2}$. (While the factor $\frac{1}{2}$ reduces the estimate of the hyperterm, it does not alter the leading term of the estimates for N_1 and N_2.) For the smallest hyperterm the latter case applies (cf equations (66) and (68)):

$$\sigma_{n,\text{as}}^{(N_0,N_1)} = (-3g)^{N_0} C_n^2 \Gamma\left(N_1 + n + \tfrac{1}{2}\right)\left[\ln\left(N_1 + n + \tfrac{1}{2}\right) + \ln 12 - \psi(n+1)\right] \times \Gamma\left(N_0 + n + \tfrac{1}{2} - N_1\right). \tag{65}$$

Note that (with $N_0 > N_1$) the signs are ultimately given by $(-1)^{N_0}$, and, except for early N_1, the hyperterms of a given subseries all have the same sign, in contrast with the alternating-sign RS terms.

First consider the parameter N_0 in equation (65) to be fixed. By using Stirling's approximation we find that for $1 \ll N_1 \ll N_0$, the hyperterms attain minimum modulus at

$$N_1 \approx \frac{N_0}{2} - \frac{1}{2\left[\ln\left(\frac{1}{2}N_0 + n + \frac{1}{2}\right) + \ln 12 - \psi(n+1)\right]}. \tag{66}$$

To find the best value of N_0, we substitute equation (66) for N_1 into equation (65), use again Stirling's approximation and get

$$\frac{N_0}{2} + n + \frac{1}{2} \approx \frac{1}{3g} + \frac{1}{2} - \frac{1}{2\left[\ln\left(\frac{1}{3g} + \frac{1}{2}\right) + \ln 12 - \psi(n+1)\right]}. \tag{67}$$

Table 5. Smallest hyperterm and corresponding values of N_0 and N_1: asymptotic estimates (equations (68)), and actual; see table 2 for notation.

		Asymptotic estimate			Actual		
n	g	$\bar{N}_0$	$\bar{N}_1$	$\left\|\sigma_{n,\mathrm{as}}^{(\bar{N}_0,\bar{N}_1)}\right\|$	N_0	N_1	$\sigma_n^{(N_0,N_1)}$
0	0.02	33.2	16.5	2.4×10^{-14}	33	16	-2.0×10^{-14}
	0.03	22.0	10.9	1.5×10^{-9}	22	10	1.1×10^{-9}
	0.04	16.5	8.1	3.7×10^{-7}	16	7	2.2×10^{-7}
	0.05	13.1	6.5	9.9×10^{-6}	13	6	-5.0×10^{-6}
1	0.02	31.1	15.5	8.0×10^{-10}	30	14	3.6×10^{-10}
	0.03	20.0	9.9	21.8×10^{-6}	19	8	-4.8×10^{-6}
	0.04	14.4	7.1	298.0×10^{-5}	12	4	31.0×10^{-5}
	0.05	11.1	5.4	508.0×10^{-4}	9	3	-28.0×10^{-4}

Table 6. Hyperenergies and accuracy (as number of significant digits) for the ground and first excited states (see table 2 for notation). The 'error' in the last column denotes the difference $E_n(g) - \Sigma_n^{(N_0,N_1)}$ between the variational eigenvalue and the corresponding partial sum. As a visual aid, the digits in the partial sums not identical with those of the variational energies have been offset by a space.

n	g	N_0	N_1	$\Sigma_n^{(N_0,N_1)}$	$-\log\left\|\sigma_n^{(N_0,N_1)}\right\|$	$-\log\|\text{error}\|$
0	0.02	33	16	0.5140864273180 467	13.7	13.5
	0.03	22	10	0.52056171 7862	9.0	8.7
	0.04	16	7	0.526733 5941	6.6	6.4
	0.05	13	6	0.53264 775	5.3	5.3
1	0.02	30	14	1.568239676 127	9.4	9.2
	0.03	19	8	1.5984 645	5.3	5.1
	0.04	12	4	1.626 285	3.5	3.3
	0.05	9	3	1.65 504	2.6	2.8

That is, we get as estimates for N_0 and N_1 (denoted by $\bar{N}_0$ and $\bar{N}_1$)

$$N_0 \approx \bar{N}_0 = 2\left(\frac{1}{3g} - n - \frac{1}{2\left[\ln\left(\frac{1}{3g}+\frac{1}{2}\right) + \ln 12 - \psi(n+1)\right]}\right). \tag{68}$$

$$N_1 \approx \bar{N}_1 = \frac{\bar{N}_0}{2} - \frac{1}{2\left[\ln\left(\frac{1}{2}\bar{N}_0 + n + \frac{1}{2}\right) + \ln 12 - \psi(n+1)\right]}. \tag{69}$$

The estimated magnitude for the smallest hyperterm is accordingly

$$\left|\sigma_{n,\mathrm{as}}^{(\bar{N}_0,\bar{N}_1)}\right| \approx \left|C_n\sqrt{2\pi}(3g)^{-n}\,\mathrm{e}^{-\frac{1}{3g}}\right|^2\left[\ln\left(\frac{1}{3g}+\frac{1}{2}\right) + \ln 12 - \psi(n+1)\right]. \tag{70}$$

The predictions of equations (66)–(70) are compared with the results of numerical calculations in tables 5 and 6. The smallest 'late' hyperterms for each g are displayed in table 6, along with the corresponding partial sums and errors.

The $n = 0$ predictions agree closely with the calculated hyperterms. For $n = 1$, the coefficient $b_1^{(2)}$ is so small that the $N_1 = 2$ term can be the smallest, and the recipe to truncate just before the smallest hyperterm would prematurely end the series. So we look for the smallest hyperterm with $N_1 \geqslant 3$. It is apparent in tables 5 and 6 that for the $n = 1$ case the indices of the smallest hyperterm as well as its magnitude are smaller than predicted. The explanation is not profound: in the $n = 1$ case the leading asymptotic estimate for $b_1^{(k)}$ is not sufficiently accurate for small k. For instance, the ratios between the exact and asymptotic values of $b_n^{(k)}$ are 0.38 ($n = 0$) and 0.05 ($n = 1$) for the small $k = 4$, and 0.95 ($n = 0$) and 0.82 ($n = 1$) for the larger $k = 50$.

4.3. Smallest second-level hyperterm and partial sum

To obtain the asymptotics of the second-level hyperterms we first focus our discussion on the double integral $T^{\{2\}}(p, q, 3g)$ given in equation (57). (Similar arguments apply to $U^{\{2\}}$ in equation (58), which has an additional logarithmic factor.) Proper treatment of the singularity in the ζ integrand would require discussion of the discontinuity of the energy in third exponentially small order, but the end result for calculation of the second-level asymptotics would be to take the principal value, which accordingly we denote by

$$I(z) = PV \int_0^\infty (3\zeta)^{-q} \frac{\mathrm{e}^{-1/3\zeta}}{\zeta - z}\,\mathrm{d}\zeta. \tag{71}$$

At $z = 0$ and $z = \infty$ we compute easily

$$I(0) = \Gamma(q) \tag{72}$$

$$I(z) \sim -\frac{1}{3z}\Gamma(q-1) \qquad (z \to \infty). \tag{73}$$

The integral $I(z)$ starts positive, ends up negative, and has a single zero approximately at $z = 1/(3q)$ where the zero of the denominator $\zeta - z$ coincides with the maximum of the numerator.

The outer z integral in $T^{\{2\}}$ is dominated by the product $(3z)^{-p}\exp[-1/(3z)]$ that has a maximum at $z = 1/(3p)$. If $p \gg q$ ($N_0 - N_1 \gg N_1 - l$), then the z integral samples mainly the early parts of $I(z)$, and via equation (72) we obtain

$$T^{\{2\}}(p, q, 3g) \sim T\left(p, \frac{1}{3g}\right)\Gamma(q) \sim \frac{1}{2}\Gamma(p)\Gamma(q) \qquad (p \gg q). \tag{74}$$

(Equation (A.5) is more appropriate for $T(p, 1/(3g))$ in the end than equation (A.3).)

On the other hand, if $p \ll q$ ($N_0 - N_1 \ll N_1 - l$), then the integral samples mainly the late part of $I(z)$, and via equation (73) we obtain

$$T^{\{2\}}(p, q, 3g) \sim -\tfrac{1}{2}\Gamma(p+1)\Gamma(q-1) \qquad (p \ll q). \tag{75}$$

The estimates (74) and (75) are opposite in sign. When $p \sim q$ ($N_0 - N_1 \sim N_1 - l$), and especially when both are near $1/(3g)$, neither estimate is valid, and $T^{\{2\}}$ crosses from positive to negative values.

The corresponding estimates for the leading asymptotics of the $T^{\{2\}}$ contribution to the second-level hyperterms are

$$\begin{aligned}(-3g)^{N_0} C_n^2 d_n^{(l)} T^{\{2\}}(p, q, 3g) &\sim \tfrac{3}{2} C_n^2 (-3g)^{N_0}\Gamma(p)\Gamma(q)\Gamma\left(n + l + \tfrac{1}{2}\right) \\ &\quad \times \left[\ln\left(l + n + \tfrac{1}{2}\right) + \ln 12 - \psi(n+1)\right]^2 \qquad (N_0 - N_1 \gg N_1 - l)\end{aligned} \tag{76}$$

and

$$\begin{aligned}(-3g)^{N_0} C_n^2 d_n^{(l)} T^{\{2\}}(p, q, 3g) &\sim -\tfrac{3}{2} C_n^2 (-3g)^{N_0}\Gamma(p+1)\Gamma(q-1)\Gamma\left(n + l + \tfrac{1}{2}\right) \\ &\quad \times \left[\ln\left(l + n + \tfrac{1}{2}\right) + \ln 12 - \psi(n+1)\right]^2 \qquad (N_0 - N_1 \ll N_1 - l).\end{aligned} \tag{77}$$

To find the values of N_0, N_1 and $N_2 = l$ for which the second-level hyperterm estimates have minimum magnitude, we borrow from the discussion relating to the hyperterms. We ignore the logarithmic factors, which have only a small effect on the location of the minimum, and find that the best value of N_2, with N_0 and N_1 fixed, is

$$N_2 = l \approx \tfrac{1}{2}N_1 \qquad N_0 - N_1 \gg N_1 - N_2 \sim \tfrac{1}{2}N_1 \tag{78}$$

or

$$N_2 = l \approx \tfrac{1}{2}(N_1 - 1) \qquad N_0 - N_1 \ll N_1 - N_2 \sim \tfrac{1}{2}N_1 + \tfrac{1}{2}. \tag{79}$$

In the first case, the best value of N_1 then minimizes

$$\Gamma\left(N_0 - N_1 + n + \tfrac{1}{2}\right)\left[\Gamma\left(\tfrac{1}{2}N_1 + n + \tfrac{1}{2}\right)\right]^2 \tag{80}$$

and has the solution

$$N_1 \approx \tfrac{2}{3}N_0. \tag{81}$$

In the second case, the best value of N_1 minimizes

$$\Gamma\left(N_0 - N_1 + 1 + n + \tfrac{1}{2}\right)\left[\Gamma\left(\tfrac{1}{2}N_1 - \tfrac{1}{2} + n + \tfrac{1}{2}\right)\right]^2 \tag{82}$$

and has the solution

$$N_1 \approx \tfrac{2}{3}N_0 + 1. \tag{83}$$

In both cases the minimum value of the gamma-function product is

$$\left[\Gamma\left(\tfrac{1}{3}N_0 + n + \tfrac{1}{2}\right)\right]^3. \tag{84}$$

It must be pointed out that these values of N_1 make equalities out of the inequalities that define the two limiting cases. We conclude that the estimate for N_1 will be more reasonable than the asymptotic estimate of the second-level hyperterm.

The optimum value of N_0 minimizes

$$(3g)^{N_0}\left[\Gamma\left(\tfrac{1}{3}N_0 + n + \tfrac{1}{2}\right)\right]^3. \tag{85}$$

As discussed in the RS and hyperterm cases, when Stirling's approximation is valid

$$N_0 \sim \frac{1}{g} - 3n. \tag{86}$$

When considering the $U^{\{2\}}$ contributions, we get estimates similar to equations (76)–(86), except that there is an extra factor of 2 on the right-hand sides of the equations equivalent to (76) and (77). But in fact inserting a factor of 2 does not take proper account of the second and third terms in equations (A.18) or additionally of the similar third set of terms in equation (56). The contributions of these slighted terms are decidedly nonuniform, being of little importance away from the region of the smallest term, but growing more significant near the smallest term. We opt for simplicity by ignoring them in estimating the smallest term, but the resulting estimate must consequently be regarded as crude.

In summary, the smallest second-level hyperterms will occur generally when

$$N_0 \sim \bar{N}_0 = \frac{1}{g} - 3n = 3\bar{N}_2 \tag{87}$$

$$N_1 \sim \bar{N}_1 = \frac{2}{3g} - 2n = 2\bar{N}_2 \tag{88}$$

$$N_2 \sim \bar{N}_2 = \frac{1}{3g} - n. \tag{89}$$

A crude order-of-magnitude estimate of the absolute value of the smallest second-level hyperterm (equivalent to three times the $T^{\{2\}}$ contribution) is given by

$$\left|\sigma_{n,\mathrm{as}}^{(\bar{N}_0,\bar{N}_1,\bar{N}_2)}\right| \approx \frac{9}{2}\left|C_n\sqrt{2\pi}(3g)^{-n}\,\mathrm{e}^{-\frac{1}{3g}}\right|^3\left[\ln\left(\frac{1}{3g}+\frac{1}{2}\right)+\ln 12-\psi(n+1)\right]^2. \qquad (90)$$

However, the smallest term will also occur near where the asymptotic sign changes, so that the actual value is less predictable than the position.

The predictions of equations (87)–(90) are compared with the results of numerical calculations in tables 7 and 8. The predicted and actual N_0, N_1, N_2 values for the smallest term in table 7 are close. The largest discrepancies are in N_2 for the highest g values when $n = 1$. The asymptotic estimate for the smallest term, however, is an order of magnitude too large for $n = 0$, and two or more orders of magnitude too large for $n = 1$. There are two main reasons for these discrepancies: the values of $b_n^{(k)}$, $c_n^{(k)}$ and $d_n^{(k)}$ for small-k are overestimated by the leading asymptotic term, especially for $n = 1$, and the smallest term is not well fit by either of the asymptotic estimates (74) or (75), since it falls in the transition between the two. At least one of the two adjacent terms ($N_2 \pm 1$), also displayed in table 7, better matches the (crude) estimate.

The energies obtained by truncation of the second-level hyperasymptotic series just before the smallest terms are displayed in table 8. The accuracy estimated as the value of the smallest term is significantly more optimistic than the true accuracy in part because the smallest term is not representative of the close-by terms in the series. These are better estimators, as shown in table 7.

Note that the ($n = 1$, $g = 0.05$) second-level hyperasymptotic energy in table 8 is worse than the first-level energy in table 6. It would appear that $g = 0.05$ is greater than the largest value at which higher-level hyperasymptotics is useful for the $n = 1$ level.

The tables 4, 6 and 8 display a slightly less than threefold increase in the number of significant figures in the second-level hypersums over the RS partial sums in the most favourable cases.

4.4. *Graphical picture of hyperasymptotic summation*

To illustrate hyperasymptotic summation for the anharmonic oscillator, we plot semilogarithmically in figures 2 and 3 the magnitudes of the RS terms, the hyperterms, and second-level hyperterms for $g = 0.02$ for the ground and first-excited states. The parameters N_0 and N_1 are taken from the 'smallest second-level hyperterm' cases given in tables 7 and 8: $n = 0, N_0 = 50, N_1 = 34, N_2 = 16$, and $n = 1, N_0 = 47, N_1 = 33, N_2 = 17$. Note how in each stage the magnitude terms first decrease in magnitude (except for some early irregularities until the asymptotic behaviour of the $b_n^{(k)}$, $c_n^{(k)}$ and $d_n^{(k)}$ settles), reach a minimum, and then increase factorially with the order. The sharpness of the minimum in the plots of the $\log\left|\sigma_n^{(N_0,N_1,l)}\right|$ versus l and which defines N_2 is apparent, compared with the rounded minima for the corresponding RS and hyperterm plots. The explanation is that in the RS, hyperterm, and away-from-the-minimum second-level hyperterm cases, the asymptotic value of each term is dominated by a product of competing gamma functions or powers (equations (62), (65), (76) and (77)) that vary smoothly with the N_i. In figures 2 and 3, however, the asymptotic estimates are not valid for the second-level term near its minimum, where it is undergoing a change in sign—i.e. passing through zero. The magnitudes of the second-level hyperterms that bracket zero are expected to be much smaller than away from the sign change. The smallest regular term for $n = 0$ is of the order of magnitude 10^{-20}, while the largest term in the partial sum ($j = 49$) is near 10^2. The smallest regular term for $n = 1$ is of the order of magnitude 10^{-13},

Table 7. Smallest second-level hyperterms and corresponding values of N_0, N_1 and N_2: asymptotic estimates and actual. The two adjacent terms ($N_2 \pm 1$) are also shown (see table 2 for notation).

		Asymptotic estimate				Actual					
n	g	$\bar{N}_0$	$\bar{N}_1$	$\bar{N}_2$	$\left\lvert\sigma_{n,\mathrm{as}}^{(\bar{N}_0,\bar{N}_1,\bar{N}_2)}\right\rvert$	N_0	N_1	N_2	$\sigma_n^{(N_0,N_1,N_2)}$	$\sigma_n^{(N_0,N_1,N_2-1)}$	$\sigma_n^{(N_0,N_1,N_2+1)}$
0	0.02	50.0	33.3	16.7	4.06×10^{-20}	50	34	16	4.07×10^{-21}	2.10×10^{-20}	-1.13×10^{-20}
	0.03	33.3	22.2	11.1	6.12×10^{-13}	34	23	10	7.76×10^{-14}	3.25×10^{-13}	-1.13×10^{-13}
	0.04	25.0	16.7	8.3	2.30×10^{-9}	25	17	7	-2.64×10^{-10}	-1.00×10^{-9}	2.87×10^{-10}
	0.05	20.0	13.3	6.7	3.15×10^{-7}	19	13	6	2.75×10^{-8}	-2.97×10^{-8}	1.09×10^{-7}
1	0.02	47.0	31.3	15.7	2.24×10^{-13}	47	33	17	2.64×10^{-16}	-4.03×10^{-14}	5.91×10^{-14}
	0.03	30.3	20.2	10.1	9.73×10^{-7}	28	19	8	7.29×10^{-10}	5.46×10^{-8}	-6.38×10^{-8}
	0.04	22.0	14.7	7.33	1.51×10^{-3}	21	14	5	7.46×10^{-8}	-3.03×10^{-5}	3.75×10^{-5}
	0.05	17.0	11.3	5.7	1.03×10^{-1}	18	12	4	1.90×10^{-5}	3.94×10^{-3}	-1.60×10^{-3}

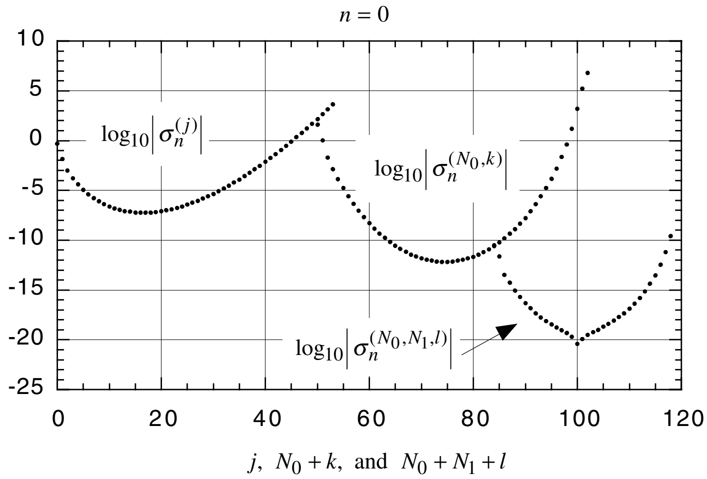

Figure 2. Semilogarithmic plot to show for the ground state the magnitudes of the RS, first-level and second-level hyperterm contributions specified by equations (29), (37) and (56). The parameter values correspond to the smallest second-level hyperterm case given in tables 7 and 8 and are $g = 0.02$, $n = 0$, $N_0 = 50$ and $N_1 = 34$. The first 50 points are the RS $\log|\sigma_0^{(j)}|$ versus j, for $j = 0, 1, \ldots, 49$. The RS values continue through $j = 53$. The hyperterm $\log|\sigma_0^{(N_0,k)}|$ values start at $j = 50$, that is, $k = j - N_0 = j - 50$, and continue through $k = 52$, which is $j = 102$. The second-level hyperterm $\log|\sigma_0^{(N_0,N_1,l)}|$ values start at $j = 84$, that is, $l = j - N_0 - N_1 = j - 50 - 34$, and continue through $l = 34$ ($j = 118$). Notice the sharp dip of the smallest second-level hyperterm at $N_2 = 16$ ($j = 100$).

Table 8. Second-level hyperenergies and accuracy (as number of significant digits) for the ground and first excited states (see table 2 for notation). The 'error' in the last column denotes the difference $E_n(g) - \Sigma_n^{(N_0,N_1,N_2)}$ between the variational eigenvalue and the corresponding partial sum. As a visual aid, the digits in the partial sums not identical with those of the variational energies have been offset by a space.

n	g	N_0	N_1	N_2	$\Sigma_n^{(N_0,N_1,N_2)}$	$-\log\left\|\sigma_n^{(N_0,N_1,N_2)}\right\|$	$-\log\|\text{error}\|$
0	0.02	50	34	16	0.514086427318015764 72 340	20.4	20.9
	0.03	34	23	10	0.5205617198730 189	13.1	13.8
	0.04	25	17	7	0.526733964 644	9.58	9.60
	0.05	19	13	6	0.5326427 739	7.56	7.72
1	0.02	47	33	17	1.5682396768049 739	15.6	13.6
	0.03	28	19	8	1.5984560 033	9.1	7.1
	0.04	21	14	5	1.626 815	7.1	4.2
	0.05	18	12	4	1.65 039	4.7	2.5

while the largest term in the partial sum ($j = 46$) is near 10^3. The cancellation of significant figures is characteristic.

5. Discussion

This paper has shown how to use the hyperasymptotic method to sum the divergent RS expansion for the perturbed harmonic oscillator to second-exponentially-small order in $\exp[-1/(3g)]$.

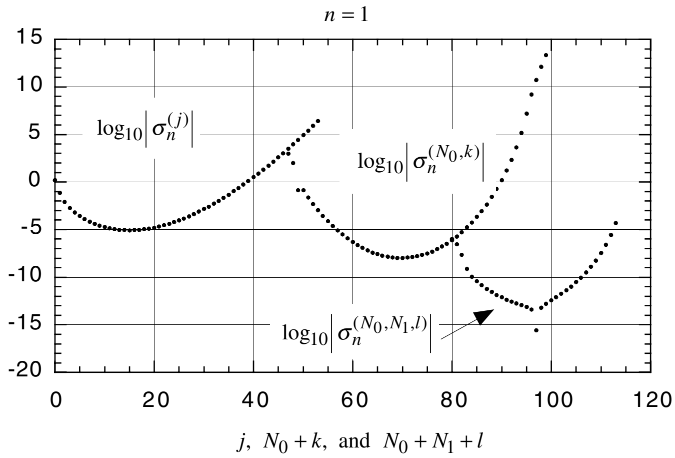

Figure 3. Semilogarithmic plot to show for the first excited state the magnitudes of the RS, first-level and second-level hyperterm contributions specified by equations (29), (37) and (56). The parameter values correspond to a smallest second-level hyperterm case given in tables 7 and 8 and are $g = 0.02$, $n = 1$, $N_0 = 47$ and $N_1 = 33$. The first 47 points are the RS $\log|\sigma_1^{(j)}|$ versus j, for $j = 0, 1, \ldots, 46$. The RS values continue through $j = 53$. The hyperterm $\log|\sigma_1^{(N_0,k)}|$ values start at $j = 47$, that is, $k = j - N_0 = j - 47$, and continue through $k = 52$, which is $j = 99$. The second-level hyperterm $\log|\sigma_1^{(N_0,N_1,l)}|$ values start at $j = 80$, that is, $l = j - N_0 - N_1 = j - 47 - 33$, and continue through $l = 33$ ($j = 113$). Notice the sharp dip of the smallest second-level hyperterm at $N_2 = 17$ ($j = 97$).

The connection between dispersion relation and hyperasymptotics is first explored via the Airy function in section 2, where the paradigm is set. Application to the perturbed oscillator is developed using the asymptotic expansion for the discontinuity in the energy calculated in the companion paper [38]. The step from first- to second-level hyperasymptotics requires a cancellation of the second-exponentially-small formally real series with the real part of the Borel sum of the first-exponentially-small formally imaginary series, which we have not proved rigorously. The first-level hyperasymptotics involve coefficients $b_n^{(k)}$ of the first-exponentially-small subseries that has been previously studied. The second-level hyperasymptotics involve coefficients $c_n^{(k)}$ and $d_n^{(k)}$ from the second-exponentially-small subseries. The early $b_n^{(k)}$ sum the late terms of the RS series. The early $c_n^{(k)}$ and $d_n^{(k)}$ sum the late terms of the $b_n^{(k)}$ series and also the direct contribution of the imaginary second-exponentially-small subseries.

To estimate the RS, hyperterm and second-level hyperterm contributions, it is necessary to have asymptotic estimates for large k for $E_n^{(k)}$, $b_n^{(k)}$, $c_n^{(k)}$ and $d_n^{(k)}$, which have been obtained in [37, 38]. Although we have omitted the formal proof, these formulae also show that in the $g \to 0$ limit the hyperasymptotic contribution approximates the first omitted RS term, while the first-level hyperasymptotic contribution approximates the first omitted hyperterm.

The estimates for the error when optimally truncated for the RS, hyperterm and second-level hyperterm series are of the order of

$$\left|C_n\sqrt{2\pi}\,(3g)^{-n}\,\mathrm{e}^{-\frac{1}{3g}}\right|^{p+1}|\ln(3g)|^p \tag{91}$$

where $p = 0, 1, 2$ denotes the highest level to which hyperasymptotics is taken. It is clear from equations (60) and (91) that for large values of g hyperasymptotics does not appear to be advantageous, but that for small values of g the accuracy can be considerable.

Balancing the formal aspects of hyperasymptotics are numerics. Formulae for evaluating the hyperterm and second-level hyperterm integrals have been derived in the appendix.

These have then been used to illustrate and support the various asymptotic estimates of optimum truncation, and to show the accuracy obtained by calculations on the ground and first excited states with $g = 0.02, 0.03, 0.04, 0.05$. All the calculations were carried out using *Mathematica*. There are three general observations: (i) in the most favourable cases treated here, accuracy is a little less than tripled in going from RS to second-level hyperasymptotics (the logarithmic factors in equations (70) and (90) reduce the gain); (ii) each successive level of hyperasymptotics requires an exponential increase in computing resources; (iii) the higher the final hyperasymptotic level, the greater the numerical cancellation in getting there. The great logarithmic gulf between largest and smallest terms (the latter characterizing the final accuracy) is manifest in figures 2 and 3. For accuracy, the advantage of *Mathematica* is to be able to specify the precision of the calculation as high as necessary.

As we mentioned briefly in the introduction, the overall behaviour of hyperasymptotic summation should be ultimately determined by the position of the 'adjacent' [2] Bender–Wu singularities in the Riemann surface of the energy [22–24] but, in the light of equations (60) and (61), hyperasymptotic summation in its present form will be useful if $1/(3g) > n$. Therefore, although all the relevant quantities have been calculated explicitly as functions of n and consequently all the states involve approximately the same computational effort, the higher the state n, the smaller the domain of validity of the procedure. (Note, however, that a fixed value of g corresponds to a much larger perturbation of a level with large n than of a level with small n, as is obvious from the larger extent of the eigenfunction for larger n.)

Our results are certainly not limited to the quartic anharmonic oscillator. We stress again that the key requirement is the existence of a dispersion relation. Therefore, the hyperasymptotic iteration can be applied straightforwardly to any perturbation of even degree gx^{2k}, and to odd perturbations gx^{2k+1} with the proviso that the complex energy then satisfies a dispersion relation in g^2, not in g. (In particular, the eigenvalues of the harmonic oscillator with cubic perturbation are real for purely imaginary values of the coupling constant g.) The method to derive the required asymptotic expansions for these perturbed oscillators has been discussed in [41, 42].

It is of course true that other numerical techniques can generate accurate eigenvalue approximations. Hyperasymptotics is however more than a procedure for obtaining exponential numerical improvements from perturbative methods [1, 2, 16–18]. Since it is based on a nonlocal incorporation of singularities, it can extend the range of validity of parameter values that can be treated, and can be used to calculate functions on and over Stokes lines into other sectors. It can also be used to discriminate between closely lying numerical solutions: exponentially small differences between two nearby solutions may grow to dominate the solutions at a later time, or for other parameter ranges. The presence and contribution of these exponentially small terms can only be discovered by calculation of the Stokes constant premultipliers. At present hyperasymptotics is the only systematic way to calculate these premultipliers to arbitrary precision [15, 18]. Finally, hyperasymptotics allows for a numerical interpretation of more advanced resurgence methods that have been applied to problems in theoretical physics [40]. The calculation of individual anharmonic eigenvalues in this paper is a further extension of this idea.

Acknowledgments

We wish to thank the Isaac Newton Institute, Cambridge University, for its sponsorship, and the Ministerio de Educación y Cultura for support under project PB98-0821.

Appendix A. Evaluation of integrals

In this appendix we discuss the evaluation of the nonlogarithmic hyperterminants (cf [26, 27]) and the new logarithmic hyperterminants.

Appendix A1. First-level hyperasymptotics: $T\left(N_0+n+\frac{1}{2}-k, \frac{1}{3g}\right)$

To compute the first-level hyperasymptotic contribution to the anharmonic oscillator energy we need to evaluate the integral (38). One formula is given in equation (18), which we repeat here with the indices relevant for the anharmonic oscillator,

$$T\left(N_0+n+\frac{1}{2}-k, \frac{1}{3g}\right) = (3g)^{-N_0-n-\frac{1}{2}+k}\,\mathrm{e}^{\frac{1}{3g}}\,\Gamma\left(N_0+n+\frac{1}{2}-k\right)\\ \times\Gamma\left(1-N_0-n-\frac{1}{2}+k, \frac{1}{3g}\right). \tag{A.1}$$

By using the gamma-function reflection formula and specializing to integer N_0+n-k, equation (A.1) can be put in a form convenient for numerical computation,

$$T\left(N_0+n+\frac{1}{2}-k, \frac{1}{3g}\right) = \frac{\pi(-1)^{N_0+n-k}\,\mathrm{e}^{\frac{1}{3g}}}{(3g)^{N_0+n+\frac{1}{2}-k}}\frac{\Gamma\left(1-N_0-n-\frac{1}{2}+k, \frac{1}{3g}\right)}{\Gamma\left(1-N_0-n-\frac{1}{2}+k\right)}. \tag{A.2}$$

The ratio of the incomplete to complete gamma function—called the 'regularized incomplete gamma function'—is a standard function available in *Mathematica*.

Appendix A2. First-level hyperasymptotics as $g\to 0$

For N_0-k fixed, we see directly from equation (38) that

$$T\left(N_0+n+\frac{1}{2}-k, \frac{1}{3g}\right) \to \Gamma\left(N_0+n+\frac{1}{2}-k\right) \qquad (g\to 0). \tag{A.3}$$

The smallest hyperterms, however, correspond to $N_0-k\approx 1/(3g)-n$, which for small g is large, and the estimate given above in equation (A.3) is not valid. Consider therefore equation (A.1) with N_0+n-k replaced by $1/(3g)$:

$$\frac{T\left(\frac{1}{3g}+\frac{1}{2}, \frac{1}{3g}\right)}{\Gamma\left(\frac{1}{3g}+\frac{1}{2}\right)} = (3g)^{-\frac{1}{3g}-\frac{1}{2}}\,\mathrm{e}^{\frac{1}{3g}}\,\Gamma\left(1-\frac{1}{3g}-\frac{1}{2}, \frac{1}{3g}\right) \xrightarrow[g\to 0]{} \frac{1}{2}. \tag{A.4}$$

The result is similar to equation (A.3), except that there is an extra factor of $1/2$:

$$T\left(N_0+n+\frac{1}{2}-k, \frac{1}{3g}\right) \sim \frac{1}{2}\Gamma\left(N_0+n+\frac{1}{2}-k\right)\\ \left(g\to 0 \text{ with } N_0+n-k\sim\frac{1}{3g}\right). \tag{A.5}$$

Appendix A3. Second-level hyperasymptotics

The integrals $T^{\{2\}}$ and $U^{\{2\}}$ do not have finite expressions in terms of simple functions [26, 27]. Furthermore, when z is on the real axis in equations (57) and (58), there is a singularity in the integrand of the inner integral.

Towards evaluating $T^{\{2\}}$ and $U^{\{2\}}$ via equations (57) and (58), consider first the inner integrations, with z real and $\epsilon > 0$:

$$\int_0^\infty (3\zeta)^{-q} \frac{\mathrm{e}^{-\frac{1}{3\zeta}}}{\zeta - (z \mp \mathrm{i}\epsilon)} \, \mathrm{d}\zeta = T\left(q, \frac{1}{3(\pm\mathrm{i}\epsilon - z)}\right) \tag{A.6}$$

$$\int_0^\infty (3\zeta)^{-q} \frac{\mathrm{e}^{-\frac{1}{3\zeta}}}{\zeta - (z \mp \mathrm{i}\epsilon)} \ln(3\zeta) \, \mathrm{d}\zeta = -\frac{\mathrm{d}}{\mathrm{d}q} T\left(q, \frac{1}{3(\pm\mathrm{i}\epsilon - z)}\right). \tag{A.7}$$

It is convenient to expand equation (A.6) in an infinite series, which can be integrated term-by-term when put into equation (57), via the following sequence of steps:

$$T(q, a) = a \int_0^\infty x^{q-1} \frac{\mathrm{e}^{-x}}{x + a} \, \mathrm{d}x \tag{A.8}$$

$$= \Gamma(q) \, \mathrm{e}^a a^q \int_a^\infty \mathrm{e}^{-t} t^{-q} \, \mathrm{d}t \tag{A.9}$$

$$= \Gamma(q) \, \mathrm{e}^a a^q \left[\Gamma(1-q) - \sum_{m=0}^\infty \frac{(-1)^m a^{m-q+1}}{m!(m-q+1)}\right] \tag{A.10}$$

$$= \frac{\pi}{\sin \pi q} \mathrm{e}^a a^q + \Gamma(q) \, \mathrm{e}^a \sum_{m=0}^\infty \frac{(-a)^{m+1}}{m!(m-q+1)}. \tag{A.11}$$

One finds for equation (A.6) that

$$T\left(q, \frac{1}{3(\pm\mathrm{i}\epsilon - z)}\right) = \frac{\pi}{\sin \pi q} [3(\pm\mathrm{i}\epsilon - z)]^{-q} \, \mathrm{e}^{1/[3(\pm\mathrm{i}\epsilon - z)]} + \Gamma(q) \, \mathrm{e}^{1/[3(\pm\mathrm{i}\epsilon - z)]} \sum_{m=0}^\infty \frac{[3(z \mp \mathrm{i}\epsilon)]^{-m-1}}{m!(m-q+1)}. \tag{A.12}$$

In the limit $\epsilon \to 0$ and z positive, one gets the sum of a purely imaginary exponentially small term and a convergent, real series (note that q is an integer plus $1/2$):

$$T\left(q, \frac{1}{3(\pm\mathrm{i}\epsilon - z)}\right) \to \mp\mathrm{i}\pi (3z)^{-q} \, \mathrm{e}^{-1/(3z)} + \Gamma(q) \, \mathrm{e}^{-1/(3z)} \sum_{m=0}^\infty \frac{(3z)^{-m-1}}{m!(m-q+1)} \quad \left(q = \text{integer} + \frac{1}{2}\right). \tag{A.13}$$

The logarithmic integral (A.7) is a derivative with respect to q. We evaluate it via equation (A.12):

$$-\frac{\mathrm{d}}{\mathrm{d}q} T\left(q, \frac{1}{3(\pm\mathrm{i}\epsilon - z)}\right) = \frac{\pi}{\sin \pi q} [3(\pm\mathrm{i}\epsilon - z)]^{-q} \, \mathrm{e}^{1/[3(\pm\mathrm{i}\epsilon - z)]} [\pi \cot \pi q + \ln[3(\pm\mathrm{i}\epsilon - z)]] - \Gamma(q) \, \mathrm{e}^{1/[3(\pm\mathrm{i}\epsilon - z)]} \sum_{m=0}^\infty \frac{[3(z \mp \mathrm{i}\epsilon)]^{-m-1}}{m!(m-q+1)} \left[\psi(q) + \frac{1}{m-q+1}\right]. \tag{A.14}$$

Again, in the limit $\epsilon \to 0$ and z positive, one gets the sum of a purely imaginary exponentially small term and a convergent, real series. But in addition, there is an exponentially small real term coming from $\ln[3(\pm i\epsilon - z)] \to \ln(3z) \pm i\pi$

$$-\frac{d}{dq}T\left(q, \frac{1}{3(\pm i\epsilon - z)}\right) \to \mp i\pi(3z)^{-q}\, e^{-1/(3z)} \ln(3z) + \pi^2(3z)^{-q}\, e^{-1/(3z)}$$
$$- \Gamma(q)\psi(q)\, e^{-1/(3z)} \sum_{m=0}^{\infty} \frac{(3z)^{-m-1}}{m!(m-q+1)} - \Gamma(q)\, e^{-1/(3z)} \sum_{m=0}^{\infty} \frac{(3z)^{-m-1}}{m!(m-q+1)^2}. \tag{A.15}$$

With equations (A.13) and (A.15) for the inner integrations, we are in a position to evaluate $T^{\{2\}}$ and $U^{\{2\}}$. For $T^{\{2\}}$ the result is a sum over the functions T that arose at the first hyperasymptotic level:

$$T^{\{2\}}(p, q, 3g) = \mp i\frac{\pi}{2^{p+q}} T\left(p+q, \frac{2}{3g}\right) + \frac{\Gamma(q)}{2^{p+1}} \sum_{m=0}^{\infty} \frac{T\left(p+m+1, \frac{2}{3g}\right)}{m!(m-q+1)2^m}. \tag{A.16}$$

Note that for large m it follows from equation (A.8) that

$$T\left(p+m+1, \frac{2}{3g}\right) \sim \frac{2}{3g}\Gamma(p+m). \tag{A.17}$$

The ratio of successive terms in the series (A.16) approaches $1/2$, and the series is convergent. The imaginary term we presume to be cancelled by the next hyperasymptotic order. We drop it from the numerical calculations; this is identical to taking the principal value of the integrals.

For $U^{\{2\}}$ the calculation is similar, except that we omit the imaginary term from the beginning. The result is the sum of a multiple of the corresponding $T^{\{2\}}$ and two additional sets of terms:

$$U^{\{2\}}(p, q, 3g)_{\mathrm{real}} = 2T^{\{2\}}(p, q, 3g)_{\mathrm{real}}\, [\psi(q) + \ln 12 - \psi(n+1)]$$
$$+ \frac{\Gamma(q)}{2^p} \sum_{m=0}^{\infty} \frac{T\left(p+m+1, \frac{2}{3g}\right)}{m!(m-q+1)^2 2^m} - 2\frac{\pi^2}{2^{p+q}} T\left(p+q, \frac{2}{3g}\right). \tag{A.18}$$

We call the reader's attention to the $c_n^{(l)}$ terms in equation (56). The last term in equation (A.18) contributes exactly -2 times the contribution that comes from $\Delta E_n^{\{2,r\}}$. We finally remark that the term $T(p+q, 2/(3g))$ has to be evaluated with equations (18) or (A.1), and not with equation (A.2), which assumes that the first argument is an integer plus $1/2$.

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