This exam has three parts and you are required to answer all questions from **Parts B and C**, and four out of five questions from **Part A**. A suggested allocation of time, would be 20 minutes for each question in **Part A**, and roughly 30 minutes for the other five. You have four hours for the exam, plus an additional 15 minutes to read it. Please be sure to define any notation that you introduce, and to be precise and to the point in your answers – think before you write!

**Part A**

1. All but two of the following statements are false in general. Identify the statements that are true, and prove them. For each of the remaining four statements, all false, give a counterexample. Of course, the simpler the counterexample, the better.

   (a) Every sequence of real numbers has a convergent subsequence.
   
   (b) Every finite set in a metric space is a closed set.
   
   (c) Any concave function from \( \mathbb{R}_+ \) to \( \mathbb{R}_+ \) is continuous, where \( \mathbb{R}_+ = \{ x \in \mathbb{R} : x \geq 0 \} \).
   
   (d) Let \( f : [a, b] \to \mathbb{R} \) be such that \( f(a) \cdot f(b) < 0 \), \( b > a \). Then there exists \( x^* \in [a, b] \) such that \( f(x^*) = 0 \).
   
   (e) Let \( A, B \subseteq \mathbb{R}^n \) be convex and nonempty. Then there exists a \( p \in \mathbb{R}^n \) and \( c \in \mathbb{R} \) such that

      (i) \( p \mathbf{x} \geq c \quad \forall \mathbf{x} \in A \),
      
      (ii) \( p \mathbf{x} \leq c \quad \forall \mathbf{x} \in B \).

   (f) Let \( S \) be a nonempty, convex, closed and bounded set in \( \mathbb{R}^n \) and \( \varphi \) an upper semicontinuous correspondence from \( S \) to itself. Then \( \varphi \) has a fixed point, i.e., there exists \( x^* \in S \) such that \( x^* \in \varphi(x^*) \).

2. All of the following statements are false; however, two of them are true but with an additional assumption. Identify these two, and provide the additional assumption under which they are true. For each of the remaining four statements, all false, give a counterexample. Again, the simpler the counterexample, the better.

   (a) A convex production set necessarily exhibits constant returns to scale.
   
   (b) Every profit maximizing production plan is technologically efficient.
   
   (c) Under a competitive equilibrium, identical agents obtain identical commodity bundles.
(d) The core allocations of a two good, two person exchange economy in which, because of communication and transactions costs, only single-person coalitions are allowed to block, constitute the entire Edgeworth box.

(e) A two-person pure exchange economy such that each person has “well-behaved” (smooth and strictly convex) preferences has a unique competitive equilibrium, and its core coincides with its set of competitive allocations.

(f) Corresponding to every Pareto optimal allocation of a pure exchange economy in which every agent has convex preferences, there exists a price such that the price allocation pair is a competitive equilibrium.

3. Consider a moral hazard problem where there are two possible actions and both the principal and the agent are risk averse. Is it necessarily true that, in order to induce the agent to take the action desired by the principal, the Pareto efficient allocation of risk bearing must be distorted? Prove your assertion. In your answer make sure to state clearly the principal’s problem.

4. Consider a decision maker, whose initial wealth is \( w \), facing the risk of losing \( x \), where \( x \) is a random variable taking values in the interval \([-w, 0]\) and is distributed according to the cumulative distribution function \( F \). The decision maker is offered an insurance policy where the premium is given by the formula \( c + pE[I] \), where \( c > 0 \) denotes that setup cost and \( I(x) \) denotes the indemnity function and \( E \) the expectations operator. Let \( u \) and \( v \) be two distinct von Neumann-Morgenstern utility functions and suppose that \( u \) displays greater risk aversion then \( v \). Show that

(a) For each of the utility functions, the optimal insurance is either full insurance or no insurance.

(b) If \( v \) chooses to take full insurance so does \( u \).

5. Consider a strategic form game in which mixed strategies are allowed.

(a) Define strictly dominated actions.

(b) Indicate whether the following statement is true or false. If true, prove that it is true, or if false, provide a counter-example.

If an action is not strictly dominated by any (pure strategy) action, then it is not strictly dominated by any mixed strategy.

**Part B**

6. Consider the following game:

(a) Define an equilibrium for this game.
(b) Characterize the equilibria.

7. A cumulative distribution function $F$ is riskier than another cumulative distribution function $G$ if it has higher variance.

(a) Define the binary relation $F$ is riskier than another cumulative distribution function $G$.

(b) Is the relation complete? Is it transitive?

(c) Is the statement true? Prove your assertion.

(d) Show that $F$ is riskier than $G$ is equivalent to the statement “every risk averse decision makers prefers $G$ over $F$.”

8. In *The Theory of Value*, Debreu presents a set of conditions under which a preference relation can be represented by a utility function. Explain what such a result means. In particular, can an *incomplete* and a *non-transitive* preference relation be represented by a utility function? In giving reasons for your answer, include precise definitions of the italicized terms. Draw the relevance, briefly and sharply, of Debreu’s result to

(a) basic results in the theory of general economic equilibrium,

(b) the definition and existence of Nash equilibrium.

Part C

9. Consider the following auction in which a seller with one item to sell faces two potential buyers whose valuations are independently distributed with uniform distribution on $[0,1]$. The auction procedure is as follows: First, buyer 1 submits a bid $b \in [0,1]$ to the seller. After observing the bid, the seller makes a take-it-or-leave-it offer $p \in [0,1]$ to buyer 2. If buyer 2 accepts, then the item is sold to buyer 2 at price $p$. If buyer 2 rejects, then the item is sold to buyer 1 at price $b$.

(a) Define an equilibrium for this game.

(b) Characterize the equilibrium.

(c) What is the seller’s expected revenue?

10. (i) Explain clearly, but briefly, the difference between an *exchange economy* and a *game*.
(ii) Consider an exchange economy with two goods and two consumers, $A$ and $B$. Both consumers have tastes parametrized by linear indifference curves, but consumer $A$ likes both commodities equally whereas the consumer $B$ prefers the second commodity twice as much as the first commodity. Consumer $A$'s initial endowment is $6, 2$ and consumer $B$'s initial endowment is $4, 6$. For this situation:

(a) Draw the Edgeworth box for this economy, taking care to label representative indifference curves for each consumer.

(b) Label the set of individually rational, Pareto optimal and the core allocations.

(c) Determine the competitive equilibrium prices and allocations.