The Johns Hopkins University  
Department of Economics  
Microeconomics Comprehensive Examination  
February 3, 2014

Instructions:

- You have 4 hours for the exam plus an additional 15 minutes to read it.
- Please answer all questions.
- PLEASE MAKE SURE TO DEFINE YOUR TERMS PRECISELY INCLUDING ANY NOTATION THAT YOU INTRODUCE.

Part A

1. 20 points

   (a) Give a statement of Kakutani’s fixed point theorem, and indicate how it is a generalization of a fixed point theorem of Brouwer. Show that Brouwer’s fixed point theorem is false if any two of the conditions (of your choice) assumed in the theorem do not hold.

   (b) Give a statement of the separating hyperplane theorem for a finite-dimensional Euclidean space. Show that its conclusion is false if any two of the conditions (of your choice) assumed in the theorem do not hold.

   (c) Versions of separating hyperplane theorems and fixed point theorems assert the existence of a price system, to use the terminology of economic applications. As instruments for the development of economic theory, are they substitutes or complements? Give reasons for your answer, and include precise definitions of the italicized terms.

2. 20 points

   (a) Give precise statements of the two fundamental theorems of welfare economics, and spell out in each case the relevance of the following assumptions on the formalization of preferences and technologies: (i) convexity, (ii) differentiability, and (iii) universality of markets. Indicate how problems having to do with adverse selection and moral hazard can be seen as a consequence of the absence of markets.

   (b) Sketch the relevance of your answer in 2a above to questions concerning the allocation of resources in a world where environmental issues cannot be ignored.

3. 40 points

   (a) The equal treatment property is informally stated as a situation in which identicals are treated identically. In the context of a pure exchange economy, give a precise and formal statement of this intuition.

   (b) Define the notion of a competitive equilibrium for a pure exchange economy. Does this solution concept exhibit the equal treatment property? Justify your answer in either case. Do the properties of preferences have any relevance for your answer?

   (c) Define the notion of a Pareto optimal allocation, and indicate whether it exhibits the equal treatment property? If so, provide a proof. If not, develop an argument as to why we should have any interest in this concept.
(d) Define the notion of a core for a pure exchange economy. Does this solution concept exhibit the equal treatment property in the context of a pure exchange economy which is replicated in the sense that there are an equal number of agents of a finite number of types? Justify your answer in either case. You may assume that the preferences of each type of agent are strictly convex.

(e) Indicate how your answer to 3d above is modified in the context of exchange economies which do not necessarily have an equal number of agents of a finite number of types.

**Part B**

4. 30 points

Two players, $A$ and $B$, simultaneously allocate their forces across $n \geq 3$ homogeneous battlefields. Player $A$ has $X_A$ units of force to distribute among the battlefields, and player $B$ has $X_B$ units, where $X_A \leq X_B$. Each player must distribute their forces without knowing the opponent's distribution. The force allocated to each battlefield must be nonnegative. If $A$ sends $x^k_A$ units and $B$ sends $x^k_B$ units to the $k$th battlefield, the player who provides the higher level of force wins battlefield $k$. In the case that the players allocate the same level of force to a battlefield, player $B$ wins that battlefield. The payoff for the whole game is the proportion of the wins on the individual battlefields.

(a) Define a pure strategy equilibrium for this game.

(b) Show that if $\frac{1}{n} X_B \geq X_A$, there exists a pure strategy equilibrium in which player $B$ wins all of the battles.

(c) Show that if $\frac{1}{n} X_B < X_A \leq X_B$, then there is no pure strategy equilibrium for this game.

(d) Define a mixed strategy equilibrium for this game.

5. 30 points

Consider the following bargaining game. There are 3 players who are trying to distribute a unit of perfectly divisible cake among themselves. The payoff to player $i$ from receiving $x_i$ at period $t$ is given by $\delta^t x_i$.

The timing of events is as follows: At date 0, player $i$ is selected as the proposer with probability $p_i$ to make a proposal on how to split the cake. Each player sequentially responds by either accepting or rejecting the proposal. If a majority accepts, the game ends and the cake is split according to the accepted proposal. Otherwise, the same procedure is repeated at date 1. If the players do not reach agreement at date 1, then the game ends and each receive a payoff of 0.

(a) Define equilibrium for this game.

(b) Assume $p_1 = 0.25$, $p_2 = 0.35$ and $p_3 = 0.4$. Show that for a sufficiently large discount factor, player 3 gets the lowest equilibrium payoff.

6. 20 points Consider the following game where $k > 0$. First, Nature moves and chooses $L$ or $R$ each with equal probability.

(a) Define weak sequential equilibrium for this game.

(b) For what range of $k$, if any, there is a separating weak sequential equilibrium in which type $L$ of player 1 plays $U$ and type $R$ plays $D$?

(c) For what range of $k$, if any, there a pooling PBE in which both types play $U$?
Part C

7. **40 points** For each of the following propositions state whether it is true or false. In either case prove your assertion.

(a) Let \( u \) and \( v \) be von Neumann-Morgenstern utility functions on the interval \([a, b]\) and let \( \mathcal{F} \) be the set of all cumulative distribution functions whose support is the same interval. Suppose that, for all \( F, G \in \mathcal{F}, \int_{a}^{b} u(x) \, dF(x) = \int_{a}^{b} u(x) \, dG(x) \) implies \( \int_{a}^{b} v(x) \, dF(x) < \int_{a}^{b} v(x) \, dG(x) \), then \( u \) displays greater absolute risk aversion than \( v \).

(b) Let \( X \) be a finite set of prizes and \( \Delta(X) \) the set of all probability distributions on \( X \). A preference relation, \( \succ \), on \( \Delta(X) \) is said to display betweenness if \( p \succ q \) implies that \( p \succ \alpha p + (1 - \alpha)q \), for all \( p \) and \( q \) in \( \alpha(X) \) and \( \alpha \in (0, 1) \). If the preference relation \( \succ \) satisfies the independence axiom of expected utility theory then it satisfies betweenness.

(c) Consider a competitive labor market with two types of agents, high and low quality. Suppose that the agent’s type is private information, education is costly, and that the cost of education is increasing, convex and inversely the marginal cost related to the agent’s quality. It is impossible for separating and pooling signaling equilibria a la Spence to exist simultaneously.

8. **40 points** Consider a principal-agent problem with hidden actions. Suppose that the principal is risk neutral, the agent is risk averse, and the utility of the agent’s "outside option" is zero. Let there be two (monetary) outcomes, \( x \), and \( x' \), where \( x' > x \), and two actions, \( a \) and \( a' \). Let \( p(a) \) and \( p(a') \) denote the probabilities of \( x' \) under \( a \) and \( a' \). Assume that the agent’s (dis)utility from taking the action \( a' \) is \(-1\) while the disutility of taking the action \( a \) is zero. Denote by \( w \) and \( w' \) the monetary payoffs to the agent if the outcomes are \( x \) and \( x' \), respectively. Formulate the principal’s problem and show that:

(a) If the principal would like to induce the agent to implement the less costly action, \( a \), then the optimal contract requires that \( w = w' \).

(b) If the principal would like to induce the agent to implement the more costly action \( a' \) then the optimal contract calls for outcome-dependent payoffs to the agent such that \( w' > w \). Moreover, the agent’s utility, \( u \), satisfies \( u(w')/u(w) = -p(a)/(1-p(a)) \).

(c) Show that if the principal is risk averse and the agent is risk neutral then the optimal contract requires that \( x' - w' = x - w \).